

Mass Equidistribution for Automorphic Forms of Cohomological Type on GL_2

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Abstract

We extend Holowinsky and Soundararajan's proof of quantum unique ergodicity for holomorphic Hecke modular forms on $SL(2, \mathbb{Z})$, by establishing it for automorphic forms of cohomological type on GL_2 over an arbitrary number field which satisfy the Ramanujan bounds. In particular, we have unconditional theorems over totally real and imaginary quadratic fields. In the totally real case we show that our result implies the equidistribution of the zero divisors of holomorphic Hecke modular forms, generalising a result of Rudnick over \mathbb{Q} .

1 Introduction

One of the central problems in the subject of quantum chaos is to understand the behaviour of high energy Laplace eigenfunctions on a Riemannian manifold M . There is an important conjecture of Rudnick and Sarnak [30] which predicts one aspect of this behaviour in the case when M is compact and negatively curved, namely that the microlocal lifts of eigenfunctions tend weakly to Liouville measure on the unit tangent bundle. This is known as the quantum unique ergodicity conjecture, and has as a corollary that the L^2 mass of eigenfunctions becomes weakly equidistributed on M . We refer the reader to [19, 20, 30, 34, 37, 40, 41] for many illuminating discussions and interesting results related to this conjecture.

In this paper we shall deal with a variant of Rudnick and Sarnak's conjecture which replaces Laplace eigenfunctions with certain modular forms. This may be described most easily in the case of the modular surface $X = SL(2, \mathbb{Z}) \backslash \mathbb{H}^2$, where the objects we shall consider are holomorphic modular forms of large weight, or equivalently sections of high tensor powers of the line bundle of holomorphic differentials on X . If f is a holomorphic modular cusp form of weight k , the analogue of the L^2 mass of f is the Petersson measure

$$\mu_f = y^k |f(z)|^2 dv,$$

where dv denotes the hyperbolic volume. The measure μ_f is invariant under $SL(2, \mathbb{Z})$, and we may suppose that f has been normalised so that it descends to a probability measure

on X . The analogue of the quantum unique ergodicity conjecture for holomorphic forms is then to show that the measures μ_f tend weakly to the hyperbolic volume as the weight of f tends to infinity. This is very much in the spirit of the original conjectures, with the Cauchy-Riemann equations replacing the Laplace operator and the weight k playing the role of the eigenvalue, and was considered in [21, 32].

There are two main differences between this conjecture and the classical form of QUE. The first is that no microlocal lift is known for holomorphic forms, so we are restricted to considering equidistribution on X rather than its unit tangent bundle, and ergodic methods may not presently be applied to this problem. The second is that the literal analogue of the conjecture fails because the space of cusp forms is large, and contains elements like Δ^k (where Δ is Ramanujan's cusp form) whose mass is not equidistributing. From a number theoretic point of view it is natural to deal with this multiplicity issue by requiring f to be a Hecke eigenform, which gives a refinement of the conjecture known as arithmetic QUE. This is a natural condition to impose, as Watson's triple product formula [39] illustrates that the generalised Riemann hypothesis would imply QUE for holomorphic Hecke eigenforms with the optimal rate of equidistribution. The first unconditional results on this conjecture were obtained by Sarnak [32], who showed that it was true for dihedral forms, and Luo and Sarnak [21], who showed that it was true for almost all eigenforms of weight at most k .

In [14, 15, 36], Holowinsky and Soundararajan established QUE for all holomorphic Hecke eigenforms on the modular surface X , or more generally any noncompact congruence hyperbolic surface. Their proof is a combination of two different approaches, one based on bounding the L value appearing in Watson's triple product formula and the other on bounding shifted convolution sums, and which complement each other in a remarkable way to produce the full result. In this paper we extend Holowinsky and Soundararajan's methods to prove QUE for holomorphic Hecke eigenforms on GL_2 over a totally real number field, or more generally for automorphic forms of cohomological type on GL_2 over an arbitrary number field and which satisfy the Ramanujan bounds. For simplicity, we assume our fields to have narrow class number one throughout the paper, but this is not essential.

We shall give a simple outline of our results here, before describing them more fully once we have introduced the required notation. First let us assume that the field F over which we are working is totally real with narrow class number one. Let \mathcal{O} be the ring of integers of F , and let $\Gamma = GL^+(2, \mathcal{O})$ be the subgroup of $GL(2, \mathcal{O})$ of elements with totally positive determinant. Fix $\nu > 0$, and let $\{f_n\}$ be a sequence of holomorphic Hecke modular forms for Γ whose weights $k_n = (k_{i,n})$ satisfy $k_{i,n} \geq k'_{j,n}$ for all i and j . Our result is:

Theorem 1. *The normalised Petersson probability measures $\mu_n = y^{k_n} |f_n(z)|^2 dv$ tend weakly to the uniform measure on $\Gamma \backslash (\mathbb{H}^2)^n$ as $k \rightarrow \infty$.*

As a consequence of theorem 1, we prove that if k is a fixed positive weight and $\{f_N\}$ a sequence of holomorphic Hecke forms of weight Nk , then the zero divisors Z_N of f_N become equidistributed on $\Gamma \backslash (\mathbb{H}^2)^n$, either as Lelong $(1, 1)$ currents or as measures defined by integration over Z_N with respect to the volume form of the induced Riemannian metric. This generalises a result of Rudnick [29] on the equidistribution of zeros of Hecke modular forms on $SL(2, \mathbb{Z})$.

The statement of the mixed case of our result is a little more involved, and for now we will give it only in the case of a Bianchi manifold $Y = \Gamma \backslash \mathbb{H}^3$, where \mathcal{O} is the ring of integers in an imaginary quadratic field F and $\Gamma = SL(2, \mathcal{O})$. Let E_d be the representation $\text{Sym}^d \otimes \overline{\text{Sym}}^d$ of $SL(2, \mathbb{C})$, and let V_d be the associated local system on Y which we equip with a certain canonical positive definite norm. The objects whose equidistribution we shall now consider may be thought of either as 1-forms in $A^1(Y, V_d)$ which are harmonic with respect to the norm on V_d and are eigenforms of the Hecke operators, or as the lowest K -types in the corresponding automorphic representations of cohomological type on $\Gamma \backslash SL(2, \mathbb{C})$.

We may define analogues of the Petersson mass using either of these viewpoints. A harmonic Hecke form $\omega \in A^1(Y, V_d)$ is a section of $T^*Y \otimes V_d$ to which we may associate the measure $\mu_\omega = \|\omega\|^2 dv$, where $\|\cdot\|$ is the tensor product of the norms on T^*Y and V_d and dv is the hyperbolic volume. Alternatively, if $\phi \in \pi$ is a vector of lowest K -type we may push the measure $|\phi|^2 dg$ from $\Gamma \backslash SL(2, \mathbb{C})$ down to Y to obtain one differing from μ_ω by a constant multiple. With this notation, we may state our result:

Theorem 2. *The measures μ_ω tend weakly to the hyperbolic measure on Y as $d \rightarrow \infty$.*

1.1 Structure of the Paper

We introduce the manifolds and automorphic forms with which we shall work in section 2, before giving the full statements of our results in section 3. We describe the structure of the proof in section 4. As our proof is a direct generalisation of the methods used by Holowinsky and Soundararajan over \mathbb{Q} , we do this by first giving an overview of their proof before explaining the modifications which must be made to extend it to a number field. Sections 5 to 7 contain the generalisation of Holowinsky's method of shifted convolution sums, and section 8 contains the extension of Soundararajan's approach of triple product identities and weak subconvexity. In section 9 we combine these two approaches to establish our main result, and in section 10 we prove the generalisation of Rudnick's theorem on the equidistribution of zero divisors of holomorphic forms. Section 11 is an appendix which contains various computations which are needed in the course of the proofs.

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2 Definitions and Notation

2.1 Arithmetic Manifolds

We begin by introducing the manifolds on which we shall work. Let F be a number field of narrow class number one with degree n and r infinite places, of which r_1 are real and r_2 are complex. Let $\mathbb{F} = F \otimes_{\mathbb{Q}} \mathbb{R}$, and \mathbb{F}^+ be the subset of totally positive elements. If \mathcal{O} is the ring of integers of F , let $\mathcal{O}^+ = \mathcal{O} \cap \mathbb{F}^+$. Define μ_+ to be the group of totally positive roots of unity in F , which is the ordinary unit group if F is totally complex and trivial

otherwise, and set $\omega_+ = |\mu_+|$. Let $G_i = GL^+(2, \mathbb{R})$ for $i \leq r_1$ and $GL(2, \mathbb{C})$ otherwise, and $G = G_1 \times \dots \times G_r = GL^+(2, \mathbb{F})$. Z_i will denote the centre of G_i , and $\overline{G}_i = G_i/Z_i$. N will denote the usual unipotent subgroup of G and \overline{G} , and A and M the maximal split and compact diagonal subgroups with lower entry equal to 1. $K = K_1 \times \dots \times K_r$ will be the maximal compact. Let $\Gamma = GL^+(2, \mathcal{O})$ be the integral matrices with totally positive determinant, and define $\Gamma_\infty = \Gamma \cap B$ and $\Gamma_U = \Gamma \cap U$.

Let $\mathbb{H}_F = \overline{G}/K$ be identified with $(\mathbb{H}^2)^{r_1} \times (\mathbb{H}^3)^{r_2}$, and introduce on it the following co-ordinates:

$$\begin{aligned} z &= (z_1, \dots, z_r), \\ z_i &= x_i + iy_i, \quad x_i, y_i \in \mathbb{R} \text{ for } i \leq r_1, \\ z_i &= x_i + jy_i, \quad x_i \in \mathbb{C}, y_i \in \mathbb{R} \text{ for } i > r_1, \\ x &= (x_1, \dots, x_r), \quad y = (y_1, \dots, y_r). \end{aligned}$$

We let

$$dv = \bigwedge_{i \leq r_1} y_i^{-2} dx_i dy_i \wedge \bigwedge_{i > r_1} \frac{y_i^{-3}}{2i} dx_i d\bar{x}_i dy_i$$

be the product of standard hyperbolic measures on \mathbb{H}_F . We define $X = \Gamma \backslash \overline{G}$ and $Y = \Gamma \backslash \mathbb{H}_F$, so that automorphic forms on GL_2/F of full level are equivalent to Hecke eigenforms on X .

Throughout the paper, we will use a multi-index notation for co-ordinates on \mathbb{H}_F and the weights of automorphic forms; for instance, if y is the co-ordinate on \mathbb{H}_F introduced above and k is an r -tuple of integers, the expression y^k will denote $\prod y_i^{k_i}$. If δ_i is defined to be 1 for $i \leq r_1$ and 2 otherwise, for any r -tuple x we denote $\prod x_i^{\delta_i}$ by Nx , and the maximum of $|x_i|$ by $\|x\|$.

2.2 Eisenstein Series

In addition to the usual complete Eisenstein series, we will work with two kinds of incomplete Eisenstein series which we term ‘pure incomplete Eisenstein series’ and ‘unipotent Eisenstein series’. To define them, we must introduce the multiplicative characters of the group $\mathbb{F}_+^\times / \mathcal{O}_+^\times$ following Hecke. Let $\epsilon_j = (\epsilon_j^1, \dots, \epsilon_j^r)$, $j = 1, \dots, r-1$ be generators of \mathcal{O}_+^\times , and define A as

$$A = \begin{pmatrix} 1/n & \log |\epsilon_1^1| & \dots & \log |\epsilon_{r-1}^1| \\ \vdots & & & \\ 1/n & \log |\epsilon_1^r| & \dots & \log |\epsilon_{r-1}^r| \end{pmatrix}$$

with inverse

$$A^{-1} = \begin{pmatrix} 1 & 1 & \dots & 2 \\ e_1^1 & e_2^1 & \dots & e_n^1 \\ \vdots & & & \\ e_1^{n-1} & e_2^{n-1} & \dots & e_n^{n-1} \end{pmatrix}.$$

(Here the first row of A^{-1} contains r_1 1's and r_2 2's.) We may now define the characters $\lambda_m(y)$ for $m \in \mathbb{Z}^{r-1}$ by the following formula:

$$\begin{aligned} \lambda_m(y) &= \prod_{p=1}^n \prod_{q=1}^{n-1} |y_p|^{2\pi i m_q e_p^q} \\ &= \exp \left(\sum_{p=1}^r \beta(m, p) \log |y_p| \right), \\ \text{where } \beta(m, p) &= 2\pi i \sum_{q=1}^{r-1} m_q e_p^q. \end{aligned} \tag{1}$$

As λ_m is invariant under the action of \mathcal{O}_+^\times on \mathbb{F}^+ , it may be extended to a Hecke character on F via the isomorphism $\mathbb{F}/\mathcal{O}^\times \simeq \mathbb{F}^+/\mathcal{O}_+^\times$.

Having defined λ_m , we may let $E(z, s, m)$ denote the usual Eisenstein series associated to the character $Ny^s \lambda_m(y)$ of the cusp of X . The pure incomplete Eisenstein series are formed by automorphising a function on $\Gamma_\infty \backslash \mathbb{H}_F$ which is invariant under U and transforms according to λ_m under the norm one elements of the diagonal. They are determined by an index $m \in \mathbb{Z}^{r-1}$ and a function $\psi \in C_0^\infty(\mathbb{R}^+)$, and are defined as

$$E(\psi, m|z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \psi(Ny(\gamma z)) \lambda_m(y(\gamma z)).$$

The unipotent Eisenstein series are formed by automorphising a function on \mathbb{H}_F which is only invariant under U . They are determined by a function $g \in C_0^\infty(\mathbb{R}_+^r)$, and defined as

$$E(g|z) = \sum_{\gamma \in \Gamma_U \backslash \Gamma} g(y(\gamma z)).$$

We note that it is less standard to form Eisenstein series by symmetrising a function over Γ_U in this way, and while these series do not play a major part in the proof, their appearance is related to the key fact that the correct way in which to generalise Holowsky's methods is by unfolding over the unipotent, as will be discussed in section 4.2.

2.3 Representation Theory of $SL(2, \mathbb{C})$

For $m \in \mathbb{N}$, let ρ_m denote the irreducible $m+1$ dimensional representation of $SU(2) \subset SL(2, \mathbb{C})$ with Hermitian inner product $\langle \cdot, \cdot \rangle$, and let \cdot^* denote the associated conjugate

linear isomorphism between ρ_m and ρ_m^* . We choose an orthonormal basis $\{v_t\}$ ($t = m, m-2, \dots, -m$) for ρ_m and dual basis $\{v_t^*\}$ for ρ_m^* , consisting of eigenvectors of M satisfying

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} v_t = e^{it\theta} v_t, \quad \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} v_t^* = e^{-it\theta} v_t^*.$$

If $r \in \mathbb{C}$ and $k \in \mathbb{Z}$, let $I_{(k,r)}$ be the representation of $SL(2, \mathbb{C})$ unitarily induced from the character

$$\chi : \begin{pmatrix} z & x \\ 0 & z^{-1} \end{pmatrix} \mapsto (z/|z|)^k |z|^{2ir}.$$

These are unitarisable for (k, r) in the set

$$U = \{(k, r) | r \in \mathbb{R}\} \cup \{(k, r) | k = 0, r \in i(-1, 1)\},$$

and two such representations $I_{(k,r)}$, $I_{(k',r')}$ are equivalent iff $(k, r) = \pm(k', r')$. Furthermore, these are all the irreducible unitary representations of $SL(2, \mathbb{C})$ other than the trivial representation. We choose a set $U' \subset U$ representing every equivalence class in U to be

$$U' = \{(k, r) | r \in (0, \infty)\} \cup \{(k, r) | r = 0, k \geq 0\} \cup \{(k, r) | k = 0, r \in i(0, 1)\}.$$

Given $\pi \in \widehat{SL(2, \mathbb{C})}$ nontrivial, we shall say π has weight k and spectral parameter r if it is isomorphic to $I_{(k,r)}$ with $(k, r) \in U'$. As we shall work on GL_2 with trivial central character, to describe the Archimedean components of our automorphic representations it will suffice to describe their restrictions to SL_2 . At complex places we shall use the parameters just introduced, and at real places we shall use the customary weight and spectral parameter.

2.4 Automorphic Forms

We shall consider QUE for automorphic forms π on GL_2/F of full level, trivial central character and cohomological type. This means that their local factors at real places are holomorphic discrete series of even weight, and the factors at complex places have spectral parameter 0. In the notation of section 2.1, these correspond to automorphic forms on X of the prescribed Archimedean type and which are eigenfunctions of the Hecke operators. We denote the weight of π by an r -tuple $k = (k_i)$, and its normalised Hecke eigenvalues by $\lambda_\pi(\mathfrak{p})$. Define ρ_k to be the representation

$$\rho_k = \bigotimes_{i \leq r_1} \chi_{k_i} \otimes \bigotimes_{i > r_1} \rho_{k_i}$$

of K , noting that in the presence of complex places K will be nonabelian and ρ_k will have dimension greater than one for most choices of weight. As ρ_k occurs as a K -type in the Archimedean component of π , there is an embedding R_π in $\text{Hom}_K(\rho_k, L^2(X))$ corresponding to π . We may associate to R_π a section F_k of the principal bundle $X \times_K \rho_k^*$ on Y , where we recall that for a representation τ of K , $X \times_K \tau$ is the quotient of $X \times \tau$ by the right K -action

$$(x, v)k = (xk, \tau(k)^{-1}v)$$

so that sections of $X \times_K \tau$ may be thought of as sections of $X \times \tau$ satisfying

$$\tau(k)v(xk) = v(x).$$

F_k may be defined by the relation $R_\pi(v)(x) = (s(x), v)$ for $v \in \rho_k$ and $x \in X$, which may be unwound to give

$$\begin{aligned} F_k(x) &= \prod_{i>r_1} (k_i + 1)^{-1/2} \sum_t R_\pi(v_t)(x) v_t^*, \\ |F_k(x)|^2 &= \prod_{i>r_1} (k_i + 1)^{-1} \sum_t |R_\pi(v_t)(x)|^2, \end{aligned}$$

where $\{v_t\}$ is a basis of M -eigenvectors for ρ_k . Note that $|F_k(x)|^2$ descends to a function on Y . Alternatively, we may define E_k to be the restriction to Γ of the representation

$$\left(\bigotimes_{i \leq r_1} \text{Sym}^{k_i-2} \right) \otimes \left(\bigotimes_{i>r_1} \text{Sym}^{k_i/2-1} \otimes \overline{\text{Sym}}^{k_i/2-1} \right)$$

of G , and let V_k the associated local system on Y , which we equip with a certain canonical positive definite norm. Then F_k may be thought of as a harmonic 1-form which represents a cohomology class in $H^1(Y, V_k)$ (this is why π is referred to as being of cohomological type). However, we will not use this point of view in this paper, and shall only refer the reader to the book of Borel and Wallach [1] where correspondences of this kind are described explicitly.

We wish to establish the equidistribution of the probability measures $|F_k|^2 dv$ on Y , in generalisation of holomorphic QUE over \mathbb{Q} . Because the K -integrals of $|R_\pi(v_t)|^2$ are independent of t , we may let $v_k \in \rho_k$ be the vector of highest weight and think of the measure $|F_k|^2 dv$ as the pushforward of $|R_\pi(v_k)|^2 dx$ from X . In the case where F is totally real, the reader may instead let f be a holomorphic Hecke eigenform with associated representation π , and let F_k be the mass function $F_k = y^{k/2} f$. In particular, the results stated in the next section may all be read with this simpler definition in mind.

To simplify the transition from Fourier expansions to shifted convolution sums in the next chapter, we will express the Fourier expansions of all our automorphic forms by sums over the ring of integers \mathcal{O} rather than the inverse different \mathcal{O}^* as follows:

$$\phi(z) = \sum_{\xi \in \mathcal{O}} a_\xi(y) e(\text{tr}(\xi \kappa x)),$$

where κ will denote a fixed totally positive generator of \mathcal{O}^* throughout. As the F_k are vector valued, it turns out that they may be expanded in Fourier series more simply by enlarging their domain \mathbb{H}_F , in a manner which we now describe. We identify \mathbb{H}_F with the subgroup NA of G in the standard Iwasawa factorisation, and let \mathbb{H}'_F be the subgroup NAM .

We then have an inclusion of \mathbb{H}_F in \mathbb{H}'_F , and we extend our hyperbolic co-ordinate system to \mathbb{H}'_F by allowing y_i to take complex values for $i > r_1$. The K -covariance of F_k means that it is determined by its values on \mathbb{H}'_F , and these determine the embedding R_π by the formula

$$R_\pi(v)(g) = (\rho(k)v, F_k(z)),$$

where $g = zk$ is the Iwasawa factorisation of g . On \mathbb{H}'_F , we may expand F_k in a Fourier series as

$$F_k(z) = \sum_{\xi > 0} a_f(\xi) \mathbf{K}_k(\xi \kappa y) e(\text{tr}(\xi \kappa x)),$$

where $\mathbf{K}_k(y) = \otimes_{i=1}^r \mathbf{K}_i(y_i)$ and the $\mathbf{K}_i(y_i)$ are defined by

$$\mathbf{K}_i(y_i) = (y_i)^{k_i/2} \exp(-2\pi y_i) \quad \text{for } i \leq r_1, \quad (2)$$

$$\mathbf{K}_i(y_i) = |y_i|^{k_i/2+1} \sum_{j=0}^{k_i} \binom{k_i}{j}^{1/2} K_{k_i/2-j}(4\pi|y_i|) e^{(k_i-2j)i\theta_i/2} v_{k_i-2j}^*, \quad i > r_1, \quad (3)$$

and θ_i is the argument of y_i . The formula for the Whittaker functions \mathbf{K}_i at complex places is taken from Jacquet-Langlands [18]. The coefficients $a_f(\xi)$ are proportional to the Hecke eigenvalues $\lambda_\pi(\xi)$,

$$a_f(\xi) = \lambda_\pi(\xi) N \xi^{-1/2} a_f(1),$$

and the first Fourier coefficient is determined by the L^2 normalisation of F_k to be

$$|a_f(1)|^2 = \prod_{i \leq r_1} \frac{(4\pi)^{k_i}}{\Gamma(k_i)} \prod_{i > r_1} \frac{(2\pi)^{k_i}}{\Gamma(k_i/2 + 1)^2} \frac{2^{7r_2-1} \pi^{r_1+3r_2}}{|D| L(1, \text{sym}^2 \pi)}. \quad (4)$$

(See section 11.2 for this calculation.)

3 Statement of Results

Our main result is theorem 3, which establishes QUE for the sections F_k under the assumption that the associated cohomological representations π satisfy the Ramanujan bound; this is known when F is totally real or imaginary quadratic, as discussed below.

Theorem 3. *If ϕ is a Hecke-Maass cusp form, we have*

$$|\langle \phi F_k, F_k \rangle| \ll_{\phi, \epsilon, \nu} (\log \|k\|)^{-1/30+\epsilon}.$$

If ϕ is a pure incomplete Eisenstein series, we have

$$\langle \phi F_k, F_k \rangle = \frac{1}{\text{Vol}(Y)} \langle \phi, 1 \rangle + O_{\phi, \epsilon, \nu}((\log \|k\|)^{-2/15+\epsilon})$$

Theorem 3 is proven by combining the following two results, which summarise the extensions of Holowinsky and Soundararajan's respective approaches to proving the equidistribution of F_k . Their statements are almost identical to those of the original theorems over \mathbb{Q} , which are recalled in section 4.1, with the only significant difference being that in the statement of theorem 4 we must impose a mild condition that all weights tend to infinity in a uniform way.

Theorem 4. *Fix an automorphic form ϕ , and suppose that there exists a $\nu > 0$ such that $k_i > \|k\|^\nu$ for all i . Define*

$$M_k(\pi) = \frac{1}{(\log \|k\|)^2 L(1, \text{sym}^2 \pi)} \prod_{N\mathfrak{p} \leq \|k\|} \left(1 + \frac{2|\lambda_\pi(\mathfrak{p})|}{N\mathfrak{p}}\right).$$

If ϕ is a Hecke-Maass cusp form, then

$$\langle \phi F_k, F_k \rangle \ll_{\phi, \epsilon, \nu} (\log \|k\|)^\epsilon M_k(\pi)^{1/2} \quad (5)$$

for any $\epsilon > 0$. If ϕ is a pure incomplete Eisenstein series then

$$\langle \phi F_k, F_k \rangle = \frac{1}{\text{Vol}(Y)} \langle \phi, 1 \rangle + O_{\phi, \epsilon, \nu}((\log \|k\|)^\epsilon M_k(\pi)^{1/2} (1 + R_k(f))) \quad (6)$$

for any $\epsilon > 0$, where

$$R_k(f) = \frac{1}{\sqrt{Nk} L(1, \text{sym}^2 \pi)} \sum_m \int_{-\infty}^{+\infty} \frac{|L(1/2 + it, \text{sym}^2 \pi \otimes \lambda_{-m})|}{(|t| + \|m\| + 1)^A} dt.$$

Theorem 5. *If ϕ is a Hecke-Maass cusp form, we have*

$$|\langle \phi F_k, F_k \rangle| \ll_{\phi, \epsilon} \frac{(\log \|k\|)^{-1/2+\epsilon}}{L(1, \text{sym}^2 \pi)}. \quad (7)$$

If $E(\frac{1}{2} + it, m, \cdot)$ is a unitary Eisenstein series, we have

$$|\langle E(\frac{1}{2} + it, m, \cdot) F_k, F_k \rangle| \ll_\epsilon (1 + |t| + \|m\|)^{2n} \frac{(\log \|k\|)^{-1+\epsilon}}{L(1, \text{sym}^2 \pi)}. \quad (8)$$

We shall prove theorem 4 in sections 5 to 7 and theorem 5 in section 8, before combining them to give our main result in section 9. The presence of these two components and the way in which they interact makes the overall proof somewhat elaborate, and so we begin by reviewing its basic outline in the case of $SL(2, \mathbb{Z})$ and giving an overview of our modifications in section 4. Our assumption that π satisfies the Ramanujan bound is needed in the proofs of both theorem 5 and 4, in the first case to establish the weak form of Ramanujan required by Soundararajan's weak subconvexity theorem, and in the second as an ingredient in bounding shifted convolution sums. It is known when F is totally real or imaginary quadratic, and so we have an unconditional theorem in these cases. In the totally real case this is derived from Deligne's theorem by Blasius in [3], while in the imaginary quadratic case this relies on deep

work of Harris, Soudry, Taylor, Berger, Harcos et al [2, 11] and requires the construction of a theta lift from GL_2/F to GSp_4/\mathbb{Q} , where complex geometry is available. The generalisation of their results to other fields with complex places is not yet established, and consequently we have no unconditional result outside totally real and imaginary quadratic fields. On the other hand, Ramanujan will hold for forms lifted from totally real subfields and so our theorem becomes unconditional if the family of cohomological forms of fixed level has the structure suggested by the results of [6] and [22], i.e. if base change and CM constructions account for all but finitely many forms.

The assumption we have made on the uniform growth of the weight is a purely technical one, and by combining the triple product identities in section 8.1 with the Lindelöf hypothesis we see that the result should still be true without it. The reason we have adopted it is so that when we come to the point in the generalisation of Holowinsky's theorem at which we apply the sieve, it will ensure that we are sieving over a rounded subset of the ring of integers rather than a narrow box.

Theorem 3 establishes QUE for any sequence of sections F_k over a totally real or imaginary quadratic field whose weights tend to infinity with the required uniformity. However, we should ask whether such a sequence exists for these fields. When F is totally real, Riemann-Roch ensures that the dimension of the space S_k of holomorphic cusp forms of weight k is $\sim Nk$, with an exact formula established by Shimizu in [33]. Over a general field, base change from \mathbb{Q} is expected to provide $\sim k$ forms of parallel weight k on a sufficiently deep congruence subgroup of Γ , where the term 'parallel' means that the weights at all places are equal as in the totally real case. In particular, for F imaginary quadratic it has been proven by Finis, Grunewald and Tirao [6] that base change produces forms of full level and so our result is not vacuous for the Bianchi manifolds. The proof may be easily modified to allow nontrivial level in any case, so by restricting to forms base changed from \mathbb{Q} (or another totally real subfield) which are known to satisfy Ramanujan, we may view it as having content over any solvable field F .

3.1 Equidistribution of Zero Currents

One consequence of QUE for holomorphic modular forms over \mathbb{Q} is that the zeros of a sequence of forms become equidistributed with respect to hyperbolic measure as $k \rightarrow \infty$, as was proven by Shiffman and Zelditch [38] for compact hyperbolic surfaces and extended to $SL(2, \mathbb{Z}) \backslash \mathbb{H}^2$ by Rudnick [29]. Using their methods, we have derived the analogous statement about the equidistribution of the zero divisors of holomorphic modular forms from our proof of holomorphic QUE. We may prove this equidistribution either in the sense of measures of integration over the (smooth parts of the) zero divisors, or in the more refined sense of Lelong (1,1)-currents, which we now describe.

We now use \mathbb{H}^n to denote the product of n copies of the upper half plane, so that the holomorphic forms f we consider live on $Y = \Gamma \backslash \mathbb{H}^n$. In higher dimensions we may replace the sum of delta measures at the zeros of f by the current of integration over its zero divisor Z_f , which is a distribution on differential forms of bidegree $(n-1, n-1)$. If $Z_f = \sum_i \text{ord}_{V_i}(f) V_i$ is the expression of Z_f as the sum of irreducible subvarieties, then

$$(Z_f, \phi) = \sum_i \text{ord}_{V_i}(f) \int_{V_i} \phi \quad (9)$$

for all smooth, compactly supported forms ϕ on $\Gamma \backslash \mathbb{H}^n$. To define these notions in the presence of torsion in Γ , we use the standard procedure of choosing $\Gamma' \subset \Gamma$ finite index and torsion free, and defining forms, subvarieties etc. on $\Gamma \backslash \mathbb{H}^n$ to be those on $\Gamma' \backslash \mathbb{H}^n$ which are invariant under Γ . Integrals such as (9) are defined to be the lifted integral on $\Gamma' \backslash \mathbb{H}^n$ divided by $|\Gamma' : \Gamma|$. We shall use $\xrightarrow{w^*}$ to denote weak* convergence of currents. With these notions in mind, we may state our result.

Theorem 6. *Fix a weight $k = (k_i)$, $k_i > 0$, and let $\{f_N\}$ be a sequence of holomorphic Hecke modular forms of weight Nk . Define*

$$\begin{aligned} \omega &= \frac{-i}{2\pi} \partial \bar{\partial} \log y^k \\ &= \frac{1}{4\pi} \sum k_i y_i^{-2} dx_i \wedge dy_i. \end{aligned}$$

If Z_N are the zero divisors of f_N , then $\frac{1}{N}Z_N \xrightarrow{w^} \omega$, i.e.*

$$\lim_{N \rightarrow \infty} \left(\frac{1}{N} Z_N, \phi \right) = \int_Y \omega \wedge \phi$$

for all continuous, compactly supported $(n-1, n-1)$ -forms ϕ . In particular, if $k = (2, \dots, 2)$ then $\frac{1}{N}Z_N \xrightarrow{w^} \omega_0$, the Kähler form of Y with the product hyperbolic metric.*

This theorem is based on ideas from complex potential theory as developed for problems in quantum chaos in [28, 29, 38]. It may be loosely interpreted as saying that not only do the (smooth parts of the) submanifolds Z_N become equidistributed as measures of integration with respect to the induced Riemannian volume, but that the directions in which their tangent subspaces lie are also becoming equidistributed. We prove theorem 6 in section 10.

4 Outline of the Proof

4.1 The Proof Over \mathbb{Q}

We begin by giving an outline of Holowinsky and Soundararajan's proof over \mathbb{Q} , as our proof over a number field runs on the same lines as theirs. Suppose f is a holomorphic Hecke eigenform of weight k on $Y = SL(2, \mathbb{Z}) \backslash \mathbb{H}^2$, with associated mass function $F_k = y^{k/2} f$. We wish to show that the normalised probability measure $\mu_f = |F_k|^2 y^{-2} dx dy$ tends weakly to hyperbolic measure $\frac{3}{\pi} y^{-2} dx dy$ as k tends to infinity, i.e. that for all $h \in C_0^\infty(X)$

$$\mu_f(h) = \int_Y h |F_k|^2 y^{-2} dx dy \rightarrow \frac{3}{\pi} \langle h, 1 \rangle.$$

In [14, 15, 35], Holowinsky and Soundararajan have established this by decomposing h in two different bases for smooth functions on X , the first a complete set of eigenfunctions for the Laplacian and the second the incomplete Poincare series P_m , defined by

$$P_m(\psi|z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} e(mx(\gamma z))\psi(y(\gamma z))$$

for $m \in \mathbb{Z}$ and $\psi \in C_0^\infty(\mathbb{R}^+)$. The chosen basis of Laplace eigenfunctions consists of the constant function, Hecke-Maass cusp forms ϕ and unitary Eisenstein series $E(\frac{1}{2} + it, \cdot)$, and the corresponding integrals which must be estimated are $\langle \phi F_k, F_k \rangle$ and $\langle E(\frac{1}{2} + it, \cdot) F_k, F_k \rangle$. These integrals may be expressed in terms of central L -values, using the classical Rankin-Selberg formula in the first case and Watson's formula in the second, and so one may hope that the theory of L functions would provide nontrivial upper bounds for them. The convex bound just fails to be of use here, however by strengthening the convex bound by a factor of $(\log C)^{-1+\epsilon}$ where C is the analytic conductor Soundararajan obtains the following result:

Theorem 7. *If ϕ is a Hecke-Maass cusp form, we have*

$$|\langle \phi F_k, F_k \rangle| \ll_{\phi, \epsilon} \frac{(\log k)^{-1/2+\epsilon}}{L(1, \text{sym}^2 f)}.$$

If $E(\frac{1}{2} + it, \cdot)$ is a unitary Eisenstein series, we have

$$|\langle E(\frac{1}{2} + it, \cdot) F_k, F_k \rangle| \ll_\epsilon (1 + |t|)^2 \frac{(\log k)^{-1+\epsilon}}{L(1, \text{sym}^2 f)}.$$

The equidistribution of μ_f would follow from theorem 7 if one knew that $L(1, \text{sym}^2 f) \gg (\log k)^{-1/2+\delta}$ for some $\delta > 0$. This is certainly expected, as it follows from the generalised Riemann hypothesis that $L(1, \text{sym}^2 f)$ is bounded below by a power of $\ln \ln k$. The best unconditional bound in this direction is due to Hoffstein and Lockhart [12], and Goldfeld, Hoffstein and Lockhart [9], who prove that $L(1, \text{sym}^2 f) \gg (\log k)^{-1}$; this is a deep result analogous to proving that there is no Siegel zero. The bound $L(1, \text{sym}^2 f) \gg (\log k)^{-1/2+\delta}$ is known unconditionally for all but K^ϵ eigenforms of weight $\leq K$ by a zero density argument, however one cannot rule out those forms with small values of $L(1, \text{sym}^2 f)$ for which Soundararajan's approach is insufficient.

Holowinsky's approach is to test μ_f against incomplete Poincare and Eisenstein series. This is equivalent to testing μ_f against Hecke-Maass cusp forms and incomplete Eisenstein series, and evaluating the inner products $\langle \phi F_k, F_k \rangle$ by regularising them with a second incomplete Eisenstein series and then unfolding. In doing this one is led to estimating the shifted convolution sums

$$\sum_{n \sim k} \lambda_f(n) \lambda_f(n + l)$$

for fixed l as $k \rightarrow \infty$, where λ_f are the automorphically normalised Hecke eigenvalues of f , and quite strikingly one is able to obtain useful bounds for these by taking absolute

values of the terms and forgoing any additive cancellation. The idea behind this is that the eigenvalues $\lambda_f(p)$ not only satisfy the Ramanujan bound $|\lambda_f(p)| \leq 2$, but are distributed in the interval $[-2, 2]$ according to Sato-Tate measure and so on average $|\lambda_f(p)|$ will be significantly smaller than 2 (we do not need to consider dihedral forms as we are working at full level). Moreover, as a typical $\lambda_f(n)$ is a product of many $\lambda_f(p)$'s this leads to a gain on average over the bound $|\lambda_f(n)| \leq \tau(n)$. This phenomenon may also be seen in the work of Elliot, Moreno and Shahidi [5] where they prove the bound

$$\sum_{n \leq x} |\tau(n)| \ll x^{13/2} (\log x)^{-1/18},$$

where τ here denotes Ramanujan's τ -function. Holowinsky uses this idea, combined with a large sieve to show that n and $n + l$ seldom both have small prime factors, to prove the following:

Theorem 8. *If λ_f are the normalised Hecke eigenvalues as above, define*

$$M_k(f) = \frac{1}{(\log k)^2 L(1, \text{sym}^2 f)} \prod_{p \leq k} \left(1 + \frac{2|\lambda_f(p)|}{p} \right).$$

If ϕ is a Hecke-Maass cusp form, we have

$$|\langle \phi F_k, F_k \rangle| \ll_{\phi, \epsilon} (\log k)^\epsilon M_k(f)^{1/2}.$$

If $E(\psi|\cdot)$ is an incomplete Eisenstein series, we have

$$|\langle E(\psi|\cdot) F_k, F_k \rangle - \frac{3}{\pi} \langle E(\psi|\cdot), 1 \rangle| \ll_{\psi, \epsilon} (\log k)^\epsilon M_k(f)^{1/2} (1 + R_k(f)),$$

where

$$R_k(f) = \frac{1}{k^{1/2} L(1, \text{sym}^2 f)} \int_{-\infty}^{\infty} \frac{|L(1/2 + it, \text{sym}^2 f)|}{(1 + |t|)^{10}} dt.$$

One can see the appeal to Sato-Tate in the quantity $M_k(f)$ appearing in theorem 8; if we only apply the bound $|\lambda_f(p)| \leq 2$ to this, one finds that $M_k(f) \ll (\ln k)^2 L(1, \text{sym}^2 f)^{-1}$ which is of no use. However, under certain natural assumptions about the distribution of $\lambda_f(p)$ it may be shown that $M_k(f)$ is small - more precisely, in [13] Holowinsky shows that if neither $L(1, \text{sym}^2 f)$ or $L(1, \text{sym}^4 f)$ are small then we have $M_k(f) \ll (\ln k)^{-\delta}$ for some $\delta > 0$. As with Soundararajan's theorem, these assumptions may also be shown to hold for almost all eigenforms using zero density estimates.

Surprisingly, while both of these approaches may fail it can be shown that together they cover all cases completely. Intuitively speaking, if $L(1, \text{sym}^2 f) < (\log k)^{-1/2+\delta}$ is small then we should have $\lambda_f(p^2) \sim -1$ for most primes $p \leq k$ (a Siegel zero type phenomenon). However, $M_k(f)$ is proven in [15] to satisfy the upper bound

$$M_k(f) \ll (\log k)^\epsilon \exp \left(- \sum_{p \leq k} \frac{(|\lambda_f(p)| - 1)^2}{p} \right), \quad (10)$$

and if $\lambda_f(p^2) \sim -1$ then $\lambda_f(p)^2 - 1 \sim -1$, so that $\lambda_f(p) \sim 0$ for most $p \leq k$ and the right hand side of (10) should be small. The precise bound Holowinsky and Soundararajan prove based on this argument is

$$M_k(f) \ll (\log k)^{1/6} (\log \log k)^{9/2} L(1, \text{sym}^2 f)^{1/2}.$$

This inequality may be used to show that if ϕ is a cusp form and $L(1, \text{sym}^2 f) < (\log k)^{-1/3-\delta}$ for some $\delta > 0$ then $M_k(f)$, and hence $\langle \phi F_k, F_k \rangle$, is small. However, if $L(1, \text{sym}^2 f) > (\log k)^{-1/3-\delta} > (\log k)^{-1/2+\delta}$ then theorem 7 shows that $\langle \phi F_k, F_k \rangle$ is small. This shows how theorems 7 and 8 complement each other in the cusp form case, and a similar relationship holds between them in the incomplete Eisenstein case.

4.2 Extension to a Number Field

We now describe the steps that must be made to generalise the method of section 4.1 to a number field. Soundararajan's approach is the easier of the two to extend, as one has the triple product formula of Ichino [16] available to generalise Watson's formula, and Soundararajan's weak subconvexity theorem is sufficiently general to also be applicable to the central L value which appears there. The only technical difficulty is in making Ichino's formula sufficiently quantitative, which requires estimating certain Archimedean integrals. The necessary computation at complex places was carried out in [23] using a result of Michel and Venkatesh appearing in [24], while at real places it may be obtained by comparison with Watson's formula. Applying weak subconvexity is then straightforward, with the only consideration being that Soundararajan's theorem is stated for L functions over \mathbb{Q} rather than a number field. However, it is easy to show that our L functions still satisfy the required hypotheses when viewed as Euler products over \mathbb{Q} . These steps are carried out in section 8.

The modifications that must be made in the case of Holowinsky's method are more involved, and we shall now describe his method in more detail before illustrating how we have adapted it in the simple case of a real quadratic field. Holowinsky's approach for $SL(2, \mathbb{Z})$ is similar to calculating the integral of $|F_k|^2$ against a Poincare series in terms of shifted convolution sums. For a Hecke-Maass form or incomplete Eisenstein series ϕ , he defines a regularised unfolding of $\langle \phi F_k, F_k \rangle$ in terms of a fixed positive $g \in C_0^\infty(\mathbb{R}^+)$ and a slowly growing parameter T by

$$I_\phi(T) = \int_{\Gamma_\infty \backslash \mathbb{H}^2} g(Ty) \phi(z) |F_k(z)|^2 d\mu. \quad (11)$$

This behaves like the integral of $\phi |F_k|^2$ over T copies of a fundamental domain for $SL(2, \mathbb{Z})$, which may be seen by taking the Mellin transform G of g and expressing (11) in terms of Eisenstein series as

$$I_\phi(T) = \frac{1}{2\pi i} \int_{(\sigma)} G(-s) T^s \int_Y E(s, z) \phi(z) |F_k(z)|^2 d\mu.$$

Shifting the contour to $\sigma = 1/2$ then gives

$$\begin{aligned} I_\phi(T) &= cT \langle \phi F_k, F_k \rangle + O(T^{1/2}), \\ \text{where } c &= \frac{3}{\pi} \langle E(g|z), 1 \rangle. \end{aligned}$$

Holowinsky then calculates $I_\phi(T)$ in a second way using the Fourier expansions of ϕ and f ,

$$\begin{aligned} \phi(z) &= \sum_l a_l(y) \exp(2\pi i l x), \\ f(z) &= \sum_{n \geq 1} a_f(n) \exp(2\pi i n z). \end{aligned}$$

Only those l with $|l| \ll T^{1+\epsilon}$ make a significant contribution, and for those l Holowinsky considers

$$\begin{aligned} S_l(T) &= \int_{\Gamma_\infty \backslash \mathbb{H}^2} g(Ty) a_l(y) \exp(2\pi i l x) |F_k(z)|^2 d\mu \\ &\ll |a_l(T^{-1})| \sum_{n \geq 1} |a_f(n) a_f(n+l)| \left(\int_0^\infty g(Ty) y^{k-2} e^{-2\pi(2n+l)y} dy \right) \end{aligned} \quad (12)$$

so that

$$I_\phi(T) = \sum_{|l| \ll T^{1+\epsilon}} S_l(T) + O(T^{1/2}).$$

When $l \neq 0$, the regularising factor $g(Ty)$ effectively truncates the sum in (12) to $n \ll Tk$, and we end up with an upper bound for $S_l(T)$ of

$$S_l(T) \ll \frac{|a_l(T^{-1})|}{kL(1, \text{sym}^2 f)} \sum_{n \leq Tk} |\lambda_f(n) \lambda_f(n+l)|.$$

The expected main term $\frac{3}{\pi} \langle \phi, 1 \rangle$ appears in $S_0(T)$, and so to prove that $\frac{3}{\pi} \langle \phi, 1 \rangle$ and $\langle \phi F_k, F_k \rangle$ are close one needs to bound the off diagonal terms $S_l(T)$ and hence $\sum_{n \leq x} |\lambda_f(n) \lambda_f(n+l)|$. Having given up additive cancellation in this sum, Holowinsky instead proceeds by using the ideas discussed in section 4.1 to show that $|\lambda_f(n) \lambda_f(n+l)|$ is small on average.

We have extended this method to work over an arbitrary number field F , with the key innovation being the way the unfolding is carried out in the presence of units. For simplicity,

we will briefly describe the method in the case of a real quadratic field $F = \mathbb{Q}(\sqrt{d})$, and f a holomorphic Hecke modular form of parallel weight (k, k) with associated automorphic representation π . Let ϕ be a Hecke-Maass cusp form, and write the Fourier expansions of f and ϕ as

$$\begin{aligned} f(z) &= \sum_{\eta > 0} a_f(\eta) \exp(2\pi i \operatorname{tr}(\eta \kappa z)), \\ \phi(z) &= \sum_{\xi \neq 0} a_\xi(y) \exp(2\pi i \operatorname{tr}(\eta \kappa x)). \end{aligned}$$

The totally positive units \mathcal{O}_+^\times of \mathcal{O} act on the terms of these expansions, and when unfolding we must do so in a way which breaks this symmetry so that the resulting shifted convolution sums are over well rounded sets in \mathcal{O} . The correct approach is to unfold to $\Gamma_U \backslash \mathbb{H}^2 \times \mathbb{H}^2 \simeq \mathbb{R}_+^2 \times (\mathbb{R}^2/\mathcal{O})$ and localise in a set of the form $B_T \times (\mathbb{R}^2/\mathcal{O})$, where B is a ball in \mathbb{R}_+^2 and we multiply it by T^{-1} in each co-ordinate to get B_T . This lets us largely ignore the units, and when we form the analogues of $S_l(T)$ it will allow us to truncate the resulting shifted convolution sum over \mathcal{O} at each place separately. We therefore define $I_\phi(T)$ as the integral

$$I_\phi(T) = \int_{\Gamma_U \backslash \mathbb{H}^2 \times \mathbb{H}^2} g(Ty) \phi(z) |F_k(z)|^2 dv, \quad (13)$$

where now we let $h \in C_0^\infty(\mathbb{R}^+)$ be a positive function and $g \in C_0^\infty(\mathbb{R}_+^2)$ be its square. We extract a main term $cT^2 \langle \phi F_k, F_k \rangle$ from this as before, by forming the symmetrised function

$$\tilde{g}(y) = \sum_{u \in \mathcal{O}_+^\times} g(uy)$$

and expanding it in the multiplicative characters of $\mathbb{R}_+^2/\mathcal{O}_+^\times$ to express $I_\phi(T)$ in terms of integrals against Eisenstein series. When we calculate $I_\phi(T)$ in terms of the Fourier expansion of ϕ it may again be shown that only those ξ with $\|\xi\| \ll T^{1+\epsilon}$ contribute, and for these we define

$$S_\xi(T) = \int_{\Gamma_U \backslash \mathbb{H}^2 \times \mathbb{H}^2} g(Ty) a_\xi(y) \exp(2\pi i \operatorname{tr}(\xi \kappa x)) |F_k(z)|^2 dv.$$

The analogue of the upper bound on $S_\xi(T)$ for $\xi \neq 0$ in terms of shifted convolutions sums is

$$S_\xi(T) \ll |a_\xi(T^{-1})| \sum_{\eta > 0} |a_f(\eta) a_f(\eta + \xi)| \int_{\mathbb{R}_+^2} g(Ty) y^{k-2} \exp(2\pi \operatorname{tr}((2\eta + \xi) \kappa y)) dy.$$

The key feature of the integral appearing here is that it factorises over the places of $\mathbb{Q}(\sqrt{d})$, and each factor depends only on the image of $2\eta + \xi$ at that place, which lets us truncate

the sum to the ball of radius k in \mathcal{O} and leaves us with bounding $\sum_{0 < \eta < k} |\lambda_f(\eta)\lambda_f(\eta + \xi)|$. This round set is well suited to the application of the large sieve for lattices in \mathbb{R}^n , and we may carry out Holowinsky's sieving approach as before by translating congruences modulo primes \mathfrak{p} of \mathcal{O} to sieve conditions in $\mathcal{O}/p\mathcal{O}$ without significant interference from the units.

We carry this method out in detail in sections 5 to 7. The proof splits into two parts, the first of which is reducing bounds on $\langle \phi F_k, F_k \rangle$ to ones on shifted convolution sums, and the second of which is bounding these sums using the large sieve. The bulk of the work lies in the first step, and we have divided it into the case of totally real fields, carried out in section 5, and the modifications which are needed in the presence of complex places which are described in section 6. The application of the large sieve is carried out in section 7.

5 Sieving for Mass Equidistribution: The Totally Real Case

In this section we prove proposition 9 below, which reduces the problem of bounding $\langle \phi F_k, F_k \rangle$ to one of bounding shifted convolution sums. We shall assume for simplicity in this section that F is totally real, so that the key modifications in the unfolding argument can be seen more clearly, and leave the treatment of complex places for section 6. We will work with holomorphic forms rather than vector valued ones, and so let f be a L^2 normalised holomorphic Hecke eigenform of weight k with associated automorphic representation π . We assume there exists $\nu > 0$ such that $k_i \geq \|k\|^\nu$ for all i .

Proposition 9. *Let $T \geq 1$ and $\epsilon > 0$. Fix $h \in C_0^\infty(\mathbb{R}^+)$ positive and let $g \in C_0^\infty(\mathbb{F}^+)$ be its n -fold product, and define $C_g = \langle E(g|z), 1 \rangle / \text{Vol}(Y)$. Fix an automorphic form ϕ with Fourier expansion*

$$\phi(z) = \sum_{\xi \in \mathcal{O}} a_\xi(y) e(\text{tr}(\xi \kappa x)).$$

If ϕ is a Hecke-Maass cusp form, then

$$\langle \phi F_k, F_k \rangle = C_g^{-1} T^{-n} \sum_{0 < \|\xi\| < T^{1+\epsilon}} S_\xi(T) + O(T^{-n/2}). \quad (14)$$

If ϕ is a pure incomplete Eisenstein series, then

$$\langle \phi F_k, F_k \rangle = \frac{1}{\text{Vol}(Y)} \langle \phi, 1 \rangle + C_g^{-1} T^{-n} \sum_{0 < \|\xi\| < T^{1+\epsilon}} S_\xi(T) + O\left(\frac{1 + R_k(f)}{T^{n/2}}\right) \quad (15)$$

with

$$R_k(f) = \frac{1}{\sqrt{Nk} L(1, \text{sym}^2 \pi)} \sum_m \int_{-\infty}^{+\infty} \frac{|L(1/2 + it, \text{sym}^2 \pi \otimes \lambda_{-m})|}{(|t| + \|m\| + 1)^A} dt. \quad (16)$$

Furthermore, we have the bound

$$S_\xi(T) \ll \frac{|a_\xi(T^{-1})|}{NkL(1, \text{sym}^2\pi)} \left(\sum_{\eta>0} |\lambda_f(\eta)\lambda_f(\eta+\xi)| \prod_{i=1}^n h\left(\frac{T(k_i-1)}{4\pi(\eta_i+\xi_i/2)}\right) + O(Nk\|k\|^{-\nu+\epsilon}T^{n+\epsilon}) \right). \quad (17)$$

The bound we shall apply to the shifted convolution sums appearing in proposition 9 is given below; it will be proven in section 7 following Holowinsky, although it should be noted that this result may also be derived from the works of Nair [26] and Nair-Tenenbaum [27].

Proposition 10. *Let λ_1 and λ_2 be multiplicative functions on \mathcal{O}^+ satisfying $|\lambda_i(\eta)| \leq \tau_m(\eta)$ for some m . For any $x = (x_i)$ sufficiently large with respect to ϵ and satisfying $x_i \geq \|x\|^\nu$, and any fixed ξ satisfying $0 < \|\xi\| \leq \|x\|^\nu$ we have*

$$\sum_{0<\eta<x} |\lambda_1(\eta)\lambda_2(\eta+\xi)| \ll \frac{\tau(\xi)Nx}{(\log|x|)^{2-\epsilon}} \prod_{N\mathfrak{p} \leq z} \left(1 + \frac{|\lambda_1(\mathfrak{p})| + |\lambda_2(\mathfrak{p})|}{N\mathfrak{p}}\right). \quad (18)$$

To deduce theorem 4 in the totally real case from propositions 9 and 10, first apply proposition 10 with $\lambda_1 = \lambda_2 = \lambda_\pi$ and $x = Tk$ to obtain

$$\sum_{\eta>0} |\lambda_f(\eta)\lambda_f(\eta+\xi)| \prod_{i=1}^n h\left(\frac{T(k_i-1)}{4\pi(\eta_i+\xi_i/2)}\right) \ll \frac{\tau(\xi)T^n Nk}{(\log\|k\|)^{2-\epsilon}} \prod_{N\mathfrak{p} \leq \|k\|} \left(1 + \frac{2|\lambda_\pi(\mathfrak{p})|}{N\mathfrak{p}}\right). \quad (19)$$

In the case of ϕ a Maass form, we substitute this into (17) and bound $|a_\xi(T^{-1})|$ by $|\rho(\xi)|T^{-n/2+\epsilon}$ using lemma 11 from section 5.1 below. As we shall choose T so that it is bounded above by any positive power of $\|k\|$, (17) then becomes

$$S_\xi(T) \ll \frac{|\rho(\xi)|\tau(\xi)T^{n/2+\epsilon}}{L(1, \text{sym}^2\pi)(\log\|k\|)^{2-\epsilon}} \prod_{N\mathfrak{p} \leq \|k\|} \left(1 + \frac{2|\lambda_\pi(\mathfrak{p})|}{N\mathfrak{p}}\right) + O(\|k\|^{-\nu+\epsilon}).$$

Substituting this into (14) and applying the Ramanujan bound on average (20), we obtain

$$\langle \phi F_k, F_k \rangle \ll \frac{T^{n/2}(T \log\|k\|)^\epsilon}{(\log\|k\|)^2 L(1, \text{sym}^2\pi)} \prod_{N\mathfrak{p} \leq \|k\|} \left(1 + \frac{2|\lambda_\pi(\mathfrak{p})|}{N\mathfrak{p}}\right) + O(T^{-n/2})$$

which gives (5) on choosing $T^n = M_k(\pi)^{-1}$. The derivation of (6) in the pure incomplete Eisenstein series case is similar.

The organisation of this section is as follows. In section 5.1 we prove some results we shall need on the Fourier coefficients of ϕ and f , and in section 5.2 we introduce the regularised unfolding integral which is the heart of our proof before using it to relate $\langle \phi F_k, F_k \rangle$ to shifted convolution sums in section 5.3.

5.1 Fourier Coefficient Calculations

In this section we present some bounds and normalisations we shall need for the Fourier coefficients of ϕ and f . If ϕ is an automorphic form on $\Gamma \backslash (\mathbb{H}^2)^n$, we may expand it in a Fourier series as

$$\phi(z) = a_0(y) + \sum_{\xi} a_{\xi}(y) e(\text{tr}(\xi \kappa x))$$

with $a_0(y) = 0$ if ϕ is a cusp form. If ϕ is a fixed Maass cusp form with spectral parameter $r = (r_i)$ then we have the expansion

$$\phi(z) = \sqrt{Ny} \sum_{\xi \neq 0} \rho(\xi) \prod_{p=1}^n K_{ir_p}(2\pi |\xi_p| \kappa_p y_p) e(\text{tr}(\xi \kappa x)),$$

where the $\rho(\xi)$ satisfy the Ramanujan bound on average, i.e.

$$\sum_{\|\xi\| \leq T} |\rho(\xi)| \ll T^n. \quad (20)$$

If ϕ is a pure incomplete Eisenstein series $E(\psi, m|z)$, we may determine its Fourier coefficients in terms of the coefficients of the complete Eisenstein series $E(s, m, z)$. The Fourier expansion of these series was calculated by Efrat [4] to be

$$E(s, m, z) = Ny^s \lambda_m(y) + \phi(s, m) Ny^{1-s} \lambda_{-m}(y) + \frac{2^n \pi^{ns}}{\sqrt{D}} Ny^{1/2} \times \\ \sum_{\xi \neq 0} N(\xi \kappa)^{s-1/2} \lambda_m(\xi \kappa) \prod_{p=1}^n \frac{K_{s+\beta(m,p)-1/2}(2\pi |\xi_p| \kappa_p y_p)}{\Gamma(s + \beta(m, p))} \frac{\sigma_{1-2s, -2m}(\xi \kappa)}{\zeta(2s, \lambda_{-2m})} e(\text{tr}(\xi \kappa x)),$$

where $\beta(m, p)$ is as in (1) and

$$\begin{aligned} \phi(s) &= \frac{\pi^{n/2}}{\sqrt{D}} \prod_{p=1}^n \frac{\Gamma(s + \beta(m, p) - 1/2)}{\Gamma(s + \beta(m, p))} \frac{\zeta(2s - 1, \lambda_{-2m})}{\zeta(2s, \lambda_{-2m})} = \frac{\theta(s - 1/2)}{\theta(s)}, \\ \theta(s) &= \pi^{-ns} D^s \prod_{p=1}^n \Gamma(s + \beta(m, p)) \zeta(2s, \lambda_{-2m}), \\ \sigma_{1-2s, -2m}(\xi \kappa) &= \sum_{\substack{(c) \\ \xi/c \in \mathcal{O}}} \frac{\lambda_{-2m}(c)}{|Nc|^{2s-1}}. \end{aligned}$$

We have

$$E(\psi, m|z) = \frac{1}{2\pi i} \int_{(2)} \Psi(-s) E(s, m, z) ds,$$

where Ψ is the Mellin transform of ψ . From this formula we may calculate the Fourier coefficients of $E(\psi, m|z)$, obtaining the expression

$$\begin{aligned} a_0(y) &= \frac{1}{2\pi i} \int_{(2)} \Psi(-s)(Ny^s \lambda_m(y) + \phi(s, m)Ny^{1-s} \lambda_{-m}(y)) ds \\ &= \psi(Ny) \lambda_m(y) + O(Ny^{-1}). \end{aligned}$$

Moving the line of integration to $\sigma = 1/2$ we obtain

$$a_0(y) = \frac{1}{\text{Vol}(Y)} \langle E(\psi, m|z), 1 \rangle + O(Ny^{1/2}),$$

where the main term is only nonzero for $m = 0$. Doing the same for nonzero ξ we obtain

$$a_\xi(y) = \frac{2^n \pi^{n/2} Ny^{1/2}}{2\pi i \sqrt{D}} \int_{-\infty}^{\infty} \pi^{nit} \Psi(-1/2 - it) N(\xi \kappa)^{it} \lambda_m(\xi \kappa) \prod_{p=1}^n \frac{K_{it+\beta(m,p)}(2\pi |\xi_p| \kappa_p y_p)}{\Gamma(1/2 + it + \beta(m,p))} \frac{\sigma_{-2it, -2m}(\xi \kappa)}{\zeta(2s, \lambda_{-2m})} dt.$$

We may apply the bound

$$K_{ir}(y) \ll |\Gamma(1/2 + ir)| \left(\frac{1 + |r|}{y} \right)^A \left(1 + \frac{1 + |r|}{y} \right)^\epsilon,$$

valid for any integer $A \geq 0$ and $\epsilon > 0$, to this to obtain

$$a_\xi(y) \ll \tau(\xi) Ny^{1/2} \|\xi y\|^{-A} \prod_{i=1}^n \left(1 + \frac{1}{\xi_i y_i} \right)^\epsilon$$

Similar bounds are valid for ϕ a Maass cusp form. The bounds for both varieties of form are summarised in the following lemma:

Lemma 11. *Let ϕ be an automorphic form on Y with Fourier series expansion*

$$\phi(z) = a_0(y) + \sum_{\xi \neq 0} a_\xi(y) e(\text{tr}(\xi \kappa x)).$$

If ϕ is a Maass cusp form, then $a_0(y) = 0$ and for $\xi \neq 0$ we have

$$a_\xi(y) \ll |\rho(\xi)| Ny^{1/2} \|\xi y\|^{-A} \prod_{i=1}^n \left(1 + \frac{1}{\xi_i y_i} \right)^\epsilon$$

for any integer $A \geq 0$ and any $\epsilon > 0$. If ϕ is a pure incomplete Eisenstein series, then

$$a_0(y) = \frac{1}{\text{Vol}(Y)} \langle \phi, 1 \rangle + O(Ny^{1/2})$$

and for $\xi \neq 0$ we have

$$a_\xi(y) \ll \tau(\xi) Ny^{1/2} \|\xi y\|^{-A} \prod_{i=1}^n \left(1 + \frac{1}{\xi_i y_i}\right)^\epsilon$$

for any integer $A \geq 0$ and any $\epsilon > 0$.

As we shall work with the fourier expansion of f rather than F_k , the L^2 normalisation of $a_f(1)$ differs slightly from the one given in section 2.4. f has the expansion

$$f(z) = \sum_{\eta > 0} a_f(\eta) e(\text{tr}(\eta \kappa z))$$

with $a_f(\eta)$ satisfying

$$a_f(\eta) = \lambda_\pi(\eta) a_f(1) \eta^{(k-1)/2},$$

and the bound $|\lambda_\pi(\xi)| \leq \tau(\xi)$ is known by the work of Blasius [3] and Deligne. The correct normalisation of $a_f(1)$ so that $\langle F_k, F_k \rangle = 1$ is

$$|a_f(1)|^2 = \kappa^k \prod_{i=1}^n \frac{(4\pi)^{k_i}}{\Gamma(k_i)} \frac{\pi^n / 2}{DL(1, \text{sym}^2 \pi)}. \quad (21)$$

5.2 The Regularised Unfolding Integral $I_\phi(T)$

In this section we construct our main object $I_\phi(T)$. By computing it asymptotically in two ways, by contour shift and then unfolding, we will obtain a link between inner products and shifted convolution sums which will prove proposition 9. Choose a positive function $h \in C_0^\infty(\mathbb{R}^+)$, and let $g \in C_0^\infty(\mathbb{F}^+)$ be its n -fold product. Define $C_g = \langle E(g|z), 1 \rangle / \text{Vol}(Y)$. Let

$$\tilde{g} = \sum_{u \in \mathcal{O}_+^\times} g(uy)$$

be the symmetrisation of g under the action of \mathcal{O}_+^\times , and let

$$G(s, m) = \int_{\mathbb{F}^+} g(y) Ny^{s-1} \lambda_m(y) dy \quad (22)$$

be the Mellin transform of \tilde{g} thought of as a function on $\mathbb{F}^+ / \mathcal{O}_+^\times$. If \mathbb{F}_+^1 denotes the multiplicative subgroup of norm 1 elements, we may use the formula of Efrat [4] for the volume of $\mathbb{F}_+^1 / \mathcal{O}_+^\times$ to invert this, obtaining

$$\tilde{g}(y) = \frac{1}{2^n \pi i R} \sum_m \int_{(\sigma)} G(-s, -m) N y^s \lambda_m(y) ds. \quad (23)$$

Let $T \geq 1$ and consider the integral

$$I_\phi(T) = \int_{\mathbb{F}^+} g(Ty) N y^{-2} \left(\int_{\mathbb{F}/\mathcal{O}} \phi(z) |F_k(z)|^2 dx \right) dy, \quad (24)$$

which may be rewritten by substituting (23) and refolding the Eisenstein series as

$$I_\phi(T) = \frac{1}{2^n \pi i R} \sum_m \int_{(\sigma)} G(-s, -m) T^{ns} \int_Y E(s, m, z) \phi(z) |F_k(z)|^2 dv ds. \quad (25)$$

We first use a contour shift to relate $I_\phi(T)$ to the inner product $\langle \phi F_k, F_k \rangle$.

Lemma 12. *For ϕ a fixed Hecke-Maass cusp form or pure incomplete Eisenstein series we have*

$$I_\phi(T) = C_g \langle \phi F_k, F_k \rangle T^n + O(T^{n/2}).$$

Proof. Starting with equation (25) and moving the contour of integration to the line $\text{Re}(s) = 1/2$, we write

$$I_\phi(T) = C_g \langle \phi F_k, F_k \rangle T^n + R_\phi(T)$$

with C_g coming from the pole of the Eisenstein series at $s = 1$ (see section 11.3 for this calculation). $R_\phi(T)$ is the remaining integral along $\text{Re}(s) = 1/2$,

$$R_\phi(T) = \int_Y p(z) \phi(z) |F_k(z)|^2 dv,$$

with

$$p(z) = \frac{1}{2^n \pi i R} \sum_m \int_{(1/2)} G(-s, -m) T^{ns} E(s, m, z) ds.$$

From the Fourier series expansion of $E(s, m, z)$ and the bound for K_{ir} we have

$$E(s, m, z) \ll N y^{1/2} + N y^{-n-1/2} (|s| + \|m\|)^{n+2} (1 + (|s| + \|m\|) N y^{-1/n})^\epsilon,$$

so that $p(z) \ll \sqrt{N y} T^{n/2}$ if $N y \gg 1$. It follows from this and the rapid decay of $\phi(z) |F_k(z)|^2$ that $R_\phi(T) \ll_{\phi, g} T^{n/2}$. □

Restating this with $I_\phi(T)$ expressed in the form (24) gives

$$C_g \langle \phi F_k, F_k \rangle T^n + O(T^{n/2}) = \int_{\mathbb{F}^+} g(Ty) N y^{-2} \left(\int_{\mathbb{F}/\mathcal{O}} \phi(z) |F_k(z)|^2 dx \right) dy, \quad (26)$$

and we shall extract shifted convolution sums from the expression on the RHS after truncating our fixed form ϕ . Recall that this had a Fourier expansion

$$\phi(z) = \sum_{\xi \in \mathcal{O}} a_{\xi}(y) e(\text{tr}(\xi \kappa x)) \quad (27)$$

with the $a_{\xi}(y)$ bounded as in lemma 11. If ϕ is a pure incomplete Eisenstein series, then we find that the contribution to $I_{\phi}(T)$ from the tail of (27) with $\|\xi\| \geq T^{1+\epsilon}$ for any $\epsilon > 0$ is bounded by

$$I_1(T) T^{-n/2+A+\epsilon'} \sum_{\|\xi\| \geq T^{1+\epsilon}} \tau(\xi) \|\xi\|^{-A+\epsilon'} \ll T^{3n/2+\epsilon(n+1-A)}$$

by the support of g and lemma 11 (which is the source of the ϵ'). Here $I_1(T)$ is our main integral with ϕ chosen to be the constant function. As a result, the contribution of these terms to $I_{\phi}(T)$ is $\ll T^{n/2}$ after choosing A sufficiently large with respect to ϵ , and a similar argument works when ϕ is a fixed cusp form. If we define ϕ^* to be the truncated function

$$\phi^*(z) = \sum_{\|\xi\| < T^{1+\epsilon}} a_{\xi}(y) e(\text{tr}(\xi \kappa x)),$$

we therefore have

$$C_g \langle \phi F_k, F_k \rangle T^n = \int_{\mathbb{F}^+} g(Ty) N y^{-2} \left(\int_{\mathbb{F}/\mathcal{O}} \phi^*(z) |F_k(z)|^2 dx \right) dy + O(T^{n/2}). \quad (28)$$

5.3 Extracting Shifted Convolution Sums

In this section we shall expand the RHS of (28) using the Fourier expansion of ϕ^* , writing

$$C_g \langle \phi F_k, F_k \rangle T^n = S_0(T) + \sum_{0 < \|\xi\| < T^{1+\epsilon}} S_{\xi}(T) + O(T^{n/2})$$

where for any $\xi \in \mathcal{O}$ we define

$$S_{\xi}(T) = \int_{\mathbb{F}^+} g(Ty) N y^{-2} \left(\int_{\mathbb{F}/\mathcal{O}} a_{\xi}(y) e(\text{tr}(\xi \kappa x)) |F_k(z)|^2 dx \right) dy.$$

Note that this definition gives us (14) of proposition 9. The aim of this section is to analyse the objects $S_{\xi}(T)$ so that when we divide through by $C_g T^n$ we have the remaining equations and bounds of proposition 9. We first note that $S_0(T) = 0$ for ϕ a cusp form and by lemma 11 we have

$$S_0(T) = \left(\frac{\langle \phi, 1 \rangle}{\text{Vol}(Y)} + O(T^{-n/2}) \right) I_1(T) \quad (29)$$

for ϕ a pure incomplete Eisenstein series. We shall treat $I_1(T)$ and $S_{\xi}(T)$ for $\xi \neq 0$ separately, beginning with $\xi \neq 0$. Squaring out $|F_k(z)|^2$ and integrating in x gives

$$S_\xi(T) = \sqrt{D} \sum_{\eta > 0} \overline{a_f(\eta)} a_f(\eta + \xi) \left(\int_{\mathbb{F}^+} g(Ty) a_\xi(y) y^{k-2} e^{-2\pi \text{tr}((2\eta + \xi)\kappa y)} dy \right).$$

As the exponentials and g are positive, this satisfies

$$S_\xi(T) \ll |a_\xi(T^{-1})| \sum_{\eta > 0} |a_f(\eta) a_f(\eta + \xi)| \left(\int_{\mathbb{F}^+} g(Ty) y^{k-2} e^{-2\pi \text{tr}((2\eta + \xi)\kappa y)} dy \right).$$

Appealing to the Mellin transform H of h and applying the normalisations of $a_f(\eta)$ and $a_f(\eta + \xi)$, we may integrate in y to obtain

$$S_\xi(T) \ll \frac{|a_\xi(T^{-1})|}{NkL(1, \text{sym}^2\pi)} \sum_{\eta > 0} |\lambda_\pi(\eta) \lambda_\pi(\eta + \xi)| \\ \times \prod_{i=1}^n \left(\frac{\sqrt{\eta_i(\eta_i + \xi_i)}}{\eta_i + \xi_i/2} \right)^{k_i-1} \frac{1}{2\pi i} \int_{(\sigma)} H(-s) \left(\frac{T}{4\pi\kappa_i(\eta_i + \xi_i/2)} \right)^s \frac{\Gamma(s + k_i - 1)}{\Gamma(k_i - 1)} ds. \quad (30)$$

Note that $\sqrt{\eta_i(\eta_i + \xi_i)} \leq \eta_i + \xi_i/2$, so that these factors may be omitted. We may simplify this expression using a lemma seen in the work of Luo and Sarnak. By [21], we have

$$\frac{\Gamma(s + k_i - 1)}{\Gamma(k_i - 1)} = (k_i - 1)^s (1 + O_{a,b}(|s| + 1)^2 k_i^{-1}), \quad (31)$$

which holds by Stirling's formula for any vertical strip $0 < a \leq \text{Re}(s) \leq b$. If we apply this to (30) we may invert the Mellin transform of h to obtain

$$S_\xi(T) \ll \frac{|a_\xi(T^{-1})|}{NkL(1, \text{sym}^2\pi)} \sum_{\eta > 0} |\lambda_\pi(\eta) \lambda_\pi(\eta + \xi)| \\ \times \prod_{i=1}^n \left(h \left(\frac{T(k_i - 1)}{4\pi(\eta_i + \xi_i/2)} \right) + O \left(k_i^\epsilon \left(\frac{T}{\eta_i + \xi_i/2} \right)^{1+\epsilon} \right) \right). \quad (32)$$

The final step in proving (17) from this is showing that when this product is expanded out, the total contribution from all the error terms is $\ll Nk \|k\|^{-\nu+\epsilon} T^{n+\epsilon}$. It is enough to consider one such term which contains 'main term' factors at the first t places and error term factors at the last $n - t$. As the factors of h provide a truncation at the first t places, the contribution from this term is bounded by

$$\ll \sum_{\substack{\eta \in \mathcal{O} \\ |\eta_i + \xi_i/2| \ll Tk_i, i \leq t}} |\lambda_\pi(\eta) \lambda_\pi(\eta + \xi)| \prod_{i > t} k_i^\epsilon \left(\frac{T}{\eta_i + \xi_i/2} \right)^{1+\epsilon}. \quad (33)$$

If we let $\tau = \eta + \xi/2$, then $\tau \in \frac{1}{2}\mathcal{O}^+$ (because we may assume η_i and $\eta_i + \xi_i$ are positive), and because $|\xi_i| \ll T^{1+\epsilon}$ we have $\tau_i + T^2 \gg \max(\eta_i, \eta_i + \xi_i)$ for all i . Therefore by Deligne's bound,

$$\lambda_\pi(\eta)\lambda_\pi(\eta + \xi) \ll N\eta^\epsilon N(\eta + \xi)^\epsilon \ll \|\tau\|^\epsilon + T^\epsilon$$

and the expression (33) may be simplified to

$$\ll \sum_{\tau_i \ll Tk_i, i \leq t} (\|\tau\|^\epsilon + T^\epsilon) \prod_{i > t} k_i^\epsilon T^{1+\epsilon} \tau_i^{-1-\epsilon}.$$

Because $\tau_i \ll Tk_i$ for $i \leq t$, this may be further reduced to

$$\ll Nk^\epsilon T^{n-t+\epsilon} \sum_{\substack{\tau_i \ll Tk_i, \\ i \leq t}} \prod_{i > t} \tau_i^{-1-\epsilon}.$$

If we project the set $\{\tau \in \frac{1}{2}\mathcal{O}^+ : \tau_i \ll Tk_i, i \leq t\}$ onto the last $n-t$ real places, we obtain a set $\mathcal{O}' \in \mathbb{R}^{n-t}$, any two of whose elements are a distance $\gg \delta = (T^t \prod_{i \leq t} k_i)^{-1/(n-t)}$ from each other and the origin. The sum above may therefore be bounded by

$$\begin{aligned} &\ll T^t \prod_{i \leq t} k_i \int_{x_i \gg \delta} \prod_{i > t} x_i^{-1-\epsilon} dx_i \\ &\ll T^{t+\epsilon} \prod_{i \leq t} k_i^{1+\epsilon}, \end{aligned}$$

so that the total contribution of our error term is $\ll Nk^\epsilon T^{n+\epsilon} \prod_{i \leq t} k_i^{1+\epsilon}$. As $t < n$, we are omitting a factor of size at least $\|k\|^\nu$ from Nk , so this is $\ll Nk\|k\|^{-\nu+\epsilon} T^{n+\epsilon}$ as required. Therefore

$$\begin{aligned} S_\xi(T) \ll \frac{1}{NkL(1, \text{sym}^2\pi)} |a_\xi(T^{-1})| &\left(\sum_{\eta > 0} |\lambda_\pi(\eta)\lambda_\pi(\eta + \xi)| \prod_{i=1}^n h\left(\frac{T(k_i - 1)}{4\pi(\eta_i + \xi_i/2)}\right) \right. \\ &\left. + O(Nk\|k\|^{-\nu+\epsilon} T^{n+\epsilon}) \right), \end{aligned}$$

which is the bound (17).

We now deal with the case $\xi = 0$. Squaring out $|F_k(z)|^2$ and integrating in x gives

$$I_1(T) = \sqrt{D} \sum_{\eta > 0} |a_f(\eta)|^2 \left(\int_{\mathbb{F}^+} g(Ty) y^{k-2} e^{-4\pi \text{tr}(\eta \kappa y)} dy \right).$$

Expressing a_f in terms of λ_π and symmetrising by the action of \mathcal{O}_+^\times , this becomes

$$\begin{aligned}
I_1(T) &= \sqrt{D}|a_f(1)|^2 \kappa^{1-k} \sum_{\eta>0} |\lambda_\pi(\eta)|^2 \left(\int_{\mathbb{F}^+} g(Ty)(\eta\kappa y)^{k-1} e^{-4\pi \text{tr}(\eta\kappa y)} dy^\times \right) \\
&= \sqrt{D}|a_f(1)|^2 \kappa^{1-k} \sum_{(\eta)>0} |\lambda_\pi(\eta)|^2 \left(\int_{\mathbb{F}^+} g(Ty) \tilde{\psi}(\eta\kappa y) dy^\times \right) \\
&= \sqrt{D}|a_f(1)|^2 \kappa^{1-k} \sum_{(\eta)>0} |\lambda_\pi(\eta)|^2 \left(\int_{\mathbb{F}^+/\mathcal{O}_+^\times} \tilde{g}(Ty) \tilde{\psi}(\eta\kappa y) dy^\times \right),
\end{aligned}$$

where

$$\tilde{g}(y) = \sum_{u \in \mathcal{O}_+^\times} g(uy), \quad \tilde{\psi}(y) = \sum_{u \in \mathcal{O}_+^\times} (uy)^{k-1} \exp(-4\pi \text{tr}(uy)).$$

If we let $G(s, m)$ be the Mellin transform of \tilde{g} as in (22), then by the Mellin inversion formula we have

$$\begin{aligned}
I_1(T) &= \frac{\sqrt{D}}{2^{n-1}R} |a_f(1)|^2 \kappa^{1-k} \sum_{(\eta)} |\lambda_\pi(\eta)|^2 \frac{1}{2\pi i} \sum_m \int_{(\sigma)} G(-s, -m) \left(\frac{T}{4\pi} \right)^{ns} \\
&\quad N(\eta\kappa)^{-s} \lambda_{-m}(\eta\kappa) \prod_{i=1}^n (4\pi)^{-k_i+1} \Gamma(s + \beta(m, i) + k_i - 1) ds.
\end{aligned}$$

Forming the L -function from the sum over η , this becomes

$$\begin{aligned}
I_1(T) &= \frac{\sqrt{D}}{2^{n-1}R} |a_f(1)|^2 \kappa^{1-k-s} \lambda_{-m}(\kappa) \frac{1}{2\pi i} \sum_m \int_{(\sigma)} G(-s, -m) \left(\frac{T}{4\pi} \right)^{ns} \\
&\quad L(s, \text{sym}^2 \pi \otimes \lambda_{-m}) \frac{L(s, \lambda_{-m})}{L(2s, \lambda_{-2m})} \prod_{i=1}^n (4\pi)^{-k_i+1} \Gamma(s + \beta(m, i) + k_i - 1) ds.
\end{aligned}$$

When we substitute the value of $|a_f(1)|^2$ and shift the line of integration to $\sigma = 1/2$, we pick up a main term from the pole at $s = 1$ which is

$$\frac{\pi^n G(-1, 0) T^n}{2D\zeta_F(2)},$$

and in section 11.3 it is shown that this agrees with the expected main term $C_g T^n$. We therefore have

$$I_1(T) = C_g T^n + E_{1/2}(T), \tag{34}$$

with

$$E_{1/2}(T) = \frac{(2\pi^2)^n N \kappa^{1/2} \lambda_{-m}(\kappa)}{R \sqrt{DL(1, \text{sym}^2 \pi)}} \frac{1}{2\pi i} \sum_m \int_{(1/2)} G(-s, -m) \left(\frac{T}{4\pi} \right)^{ns} \\ L(s, \text{sym}^2 \pi \otimes \lambda_{-m}) \frac{L(s, \lambda_{-m})}{L(2s, \lambda_{-2m})} \prod_{i=1}^n \frac{\Gamma(s + \beta(m, i) + k_i - 1)}{(k_i - 1) \Gamma(k_i - 1)} ds. \quad (35)$$

If we apply the Luo-Sarnak lemma to 35 in the form

$$\left| \frac{\Gamma(s + \beta(m, i) + k_i - 1)}{\Gamma(k_i - 1)} \right| \ll (k_i - 1)^{\text{Re}(s)} (1 + |s| + \|m\|)^2,$$

together with the rapid decay of $G(s, m)$, the convex bound for $L(\lambda_{-m}, s)$ and any lower bound of the form $L(\lambda_{-2m}, 1 + it) \gg (|t| + \|m\| + 1)^{-A}$, we obtain

$$E_{1/2}(T) \ll \left(\frac{T^n}{Nk} \right)^{1/2} \frac{1}{L(1, \text{sym}^2 \pi)} \sum_m \int_{-\infty}^{+\infty} \frac{|L(1/2 + it, \text{sym}^2 \pi \otimes \lambda_{-m})|}{(|t| + \|m\| + 1)^A} ds. \quad (36)$$

The asymptotic (15) and bound (16) for the error now follow by combining (36), (34) and (29), which completes the proof of proposition 9.

6 Sieving for Mass Equidistribution: The Mixed Case

In this section we generalise proposition 9 to allow complex places of F . As we may no longer talk about holomorphic forms we now let F_k be a vector valued cohomological form with associated automorphic representation π , and as before assume the existence of a $\nu > 0$ such that $k_i \geq \|k\|^\nu$ for all i . Our bound for $\langle \phi F_k, F_k \rangle$ in terms of shifted convolution sums is as follows.

Proposition 13. *Let $T \geq 1$ and $\epsilon > 0$. Fix $h \in C_0^\infty(\mathbb{R}^+)$ positive and let $g \in C_0^\infty(\mathbb{R}_+^r)$ be its r -fold product, and define $C_g = \langle E(g|z), 1 \rangle / \text{Vol}(Y)$. Let J be the set of r_2 -tuples $J = \{(j_{r_1+1}, \dots, j_r) | 0 \leq j_i \leq k_i\}$. Fix an automorphic form ϕ with Fourier expansion*

$$\phi(z) = \sum_{\xi \in \mathcal{O}} a_\xi(y) e(\text{tr}(\xi \kappa x)).$$

If ϕ is a Hecke-Maass cusp form, then

$$\langle \phi F_k, F_k \rangle = c^{-1} T^{-n} \sum_{0 < \|\xi\| < T^{1+\epsilon}} S_\xi(T) + O(T^{-n/2}). \quad (37)$$

If ϕ is a pure incomplete Eisenstein series, then

$$\langle \phi F_k, F_k \rangle = \frac{1}{\text{Vol}(Y)} \langle \phi, 1 \rangle + c^{-1} T^{-n} \sum_{0 < \|\xi\| < T^{1+\epsilon}} S_\xi(T) + O\left(\frac{1 + R_k(f)}{T^{n/2}}\right) \quad (38)$$

with

$$R_k(f) = \frac{1}{\sqrt{Nk}L(1, \text{sym}^2\pi)} \sum_m \int_{-\infty}^{+\infty} \frac{|L(1/2 + it, \text{sym}^2\pi \otimes \lambda_{-m})|}{(|t| + \|m\| + 1)^A} dt. \quad (39)$$

Furthermore, we have the bound

$$S_\xi(T) \ll \frac{|a_\xi(T^{-1})|}{NkL(1, \text{sym}^2\pi)} \left(\sum_{j \in J} A_{\xi,j} + O(Nk\|k\|^{-\nu+\epsilon} T^{n+\epsilon}) \right), \quad (40)$$

where

$$A_{\xi,j}(T) \ll \left(\sum_{\eta > 0} |\lambda_\pi(\eta) \lambda_\pi(\eta + \xi)| \prod_{i \leq r_1} \frac{1}{k_i} h\left(\frac{T(k_i - 1)}{4\pi|\eta_i \kappa_i|}\right) \right. \\ \left. \times \prod_{i > r_1} \frac{1}{k_i(k_i - j_i)j_i} h\left(\frac{T\sqrt{j_i(k_i - j_i)}}{2\pi|\eta_i \kappa_i|}\right) \right). \quad (41)$$

Theorem 4 follows from combining this with proposition 10 as in the totally real case. Most of the added difficulty in the proof of proposition 13 comes from the changes to the Fourier expansion of F_k in the presence of complex places. The first difference is that F_k is vector valued, which is the source of the summation over J in (40). The second is that its Fourier coefficients contain Bessel functions, and so multiple Bessel integrals appear when we bound $S_\xi(T)$ in terms of shifted convolution sums. Section 6.1 below contains the Fourier expansions and L^2 normalisations of the automorphic forms on Y we shall work with, as well as the revised definition of $I_\phi(T)$ and its expression in terms of the shifted convolution integrals $S_\xi(T)$. The remainder of the proof of proposition 13 will then lie in analysing $S_\xi(T)$, which we do in section 6.2 in the case $\xi \neq 0$ and in section 6.3 in the case $\xi = 0$.

6.1 Revision of Basic Definitions

The bounds we shall use for the Fourier coefficients of Maass forms and Eisenstein series are essentially unchanged from the totally real case. If ϕ is a Maass cusp form with spectral parameter $r = (r_i)$ then we have the expansion

$$\phi(z) = \sqrt{Ny} \sum_{\xi \neq 0} \rho(\xi) \prod_{p=1}^r K_{ir_p}(2\pi\delta_p|\xi_p \kappa_p|y_p) e(\text{tr}(\xi \kappa x))$$

where the $\rho(\xi)$ satisfy the Ramanujan bound on average, i.e.

$$\sum_{\|\xi\| \leq T} |\rho(\xi)| \ll T^n. \quad (42)$$

The coefficients of a pure incomplete Eisenstein series may again be expressed in terms of those of complete Eisenstein series $E(s, m, z)$. The Fourier expansion of these is computed in section 11.1, following Efrat in the totally real case [4], and is

$$E(s, m, z) = Ny^s \lambda_m(y) + \phi(s, m) Ny^{1-s} \lambda_{-m}(y) + \frac{2^r \pi^{ns-r_2}}{\sqrt{|D|}} \sqrt{Ny} \times \\ \sum_{\xi \neq 0} N(\delta \xi \kappa)^{s-1/2} \lambda_m(\delta \xi \kappa) \prod_{p=1}^r \frac{K_{\delta_p(s-1/2)+\beta(m,p)}(2\pi \delta_p |\xi_p \kappa_p| y_p)}{\Gamma(\delta_p s + \beta(m, p))} \frac{\sigma_{1-2s, -2m}(\xi \kappa)}{\zeta(2s, \lambda_{-2m})} e(\text{tr}(\xi \kappa x)),$$

where $\beta(m, p)$ is as in (1) and

$$\begin{aligned} \phi(s) &= \frac{\pi^{n/2}}{\sqrt{|D|}} \prod_{p \leq r_1} \frac{\Gamma(s + \beta(m, p) - 1/2)}{\Gamma(s + \beta(m, p))} \prod_{p > r_1} \frac{2}{2s + \beta(m, p) - 1} \frac{\zeta(2s - 1, \lambda_{-2m})}{\zeta(2s, \lambda_{-2m})} \\ &= \frac{\theta(s - 1/2)}{\theta(s)}, \\ \theta(s) &= |D|^s \pi^{-ns} \prod_{p \leq r_1} \Gamma(s + \beta(m, p)) \prod_{p > r_1} 2^{-2s} \Gamma(2s + \beta(m, p)) \zeta(2s, \lambda_{-2m}), \\ \sigma_{1-2s, -2m}(\xi \kappa) &= \sum_{\substack{(c) \\ \xi/c \in \mathcal{O}}} \frac{\lambda_{-2m}(c)}{|Nc|^{2s-1}}. \end{aligned}$$

By Mellin inversion we again have the two asymptotics

$$\begin{aligned} a_0(y) &= \psi(Ny) \lambda_m(y) + O(Ny^{-1}), \\ &= \frac{1}{\text{Vol}(Y)} \langle E(\psi, m|z), 1 \rangle + O(Ny^{1/2}) \end{aligned} \quad (43)$$

for the zeroth Fourier coefficient of $E(\psi, m|z)$ as Ny tends to 0 and infinity. The bounds on the nonzero coefficients of Hecke-Maass cusp forms and pure incomplete Eisenstein series are also unchanged, and so lemma 11 of section 5.1 continues to hold. We recall the formula for the Fourier expansion of the vector valued function F_k in \mathbb{H}'_F :

$$F_k(z) = \sum_{\eta > 0} a_f(\eta) \mathbf{K}_k(\eta \kappa y) e(\text{tr}(\eta \kappa x)),$$

where $\mathbf{K}_k(y)$ are as in (2) and (3). The coefficients $a_f(\eta)$ satisfy the proportionality relation $a_f(\eta) = \lambda_\pi(\eta) N \eta^{-1/2} a_f(1)$, where $a_f(1)$ is given by (4). As before, we assume the

Ramanujan bound $|\lambda_\pi(\eta)| \leq \tau(\eta)$; see the discussion of section 3 for the circumstances under which this is known.

$I_\phi(T)$ is still a regularised unfolding integral over Γ_U , and when constructing it we bear in mind that $\mathbb{H}_F \simeq \mathbb{R}_+^r \times \mathbb{F}$, and $\Gamma_U \backslash \mathbb{H}_F \simeq \mathbb{R}_+^r \times (\mathbb{F}/\mathcal{O})$. To define $I_\phi(T)$ we therefore choose a positive function $h \in C_0^\infty(\mathbb{R}^+)$ and let $g \in C_0^\infty(\mathbb{R}_+^r)$ be its r -fold product. Let

$$\tilde{g} = \sum_{u \in \mathcal{O}_+^\times} g(|u|y)$$

be the symmetrisation of g under the action of \mathcal{O}_+^\times , and let

$$G(s, m) = \int_{\mathbb{R}_+^r} g(y) N y^s \lambda_m(y) dy^\times \quad (44)$$

be the Mellin transform of \tilde{g} thought of as a function on $\mathbb{R}_+^r / \mathcal{O}_+^\times$. After calculating the volume of $\mathbb{R}_+^r / \mathcal{O}_+^\times$ with the restriction of hyperbolic measure as in Efrat [4], we may invert to obtain

$$\tilde{g}(y) = \frac{1}{2\pi i V_c} \sum_m \int_{(\sigma)} G(-s, -m) N y^s \lambda_m(y) ds, \quad (45)$$

where V_c is the volume of $\mathbb{R}_+^r / \mathcal{O}_+^\times$ and is equal to $2^{r_1 - r_2 - 1 + \delta_{0r_1}} R$ (see section 11.3 for this calculation). We now define $I_\phi(T)$ to be

$$I_\phi(T) = \int_{\mathbb{R}_+^r} g(Ty) N y^{-1} \left(\int_{\mathbb{F}/\mathcal{O}} \phi(z) |F_k(z)|^2 dx \right) dy^\times. \quad (46)$$

To rewrite this in terms of integrals against Eisenstein series, we symmetrise over \mathcal{O}_+^\times and substitute (45), giving

$$\begin{aligned} I_\phi(T) &= \frac{1}{2\pi i V_c} \sum_m \int_{(\sigma)} G(-s, -m) T^{ns} \int_{\mathbb{R}_+^r / \mathcal{O}_+^\times} N y^{s-1} \lambda_m(y) \\ &\quad \int_{\mathbb{F}/\mathcal{O}} \phi(z) |F_k(z)|^2 dx dy^\times ds \\ &= \frac{\omega_+}{2\pi i V_c} \sum_m \int_{(\sigma)} G(-s, -m) T^{ns} \int_{\Gamma_\infty \backslash \mathbb{H}_F} N y^s \lambda_m(y) \phi(z) |F_k(z)|^2 dv ds \\ &= \frac{\omega_+}{2\pi i V_c} \sum_m \int_{(\sigma)} G(-s, -m) T^{ns} \int_Y E(s, m, z) \phi(z) |F_k(z)|^2 dv ds. \end{aligned}$$

On shifting the line of integration to $\sigma = 1/2$, we have the asymptotic

$$I_\phi(T) = C_g \langle \phi F_k, F_k \rangle T^n + O(T^{n/2}). \quad (47)$$

(See section 11.3 for the verification that the residue at $s = 1$ is correct.) Comparing this with the form (46) of $I_\phi(T)$ and truncating the Fourier expansion of ϕ to those terms with $\|\xi\| \ll T^{1+\epsilon}$ we arrive at the equation

$$c\langle \phi F_k, F_k \rangle T^n + O(T^{n/2}) = \int_{\mathbb{R}_+^r} g(Ty) Ny^{-1} \left(\int_{\mathbb{F}/\mathcal{O}} \phi^*(z) |F_k(z)|^2 dx \right) dy^\times \quad (48)$$

where $\phi^*(z) = \sum_{\|\xi\| < T^{1+\epsilon}} a_\xi(y) e(\text{tr}(\xi \kappa x)),$

which is the starting point for our analysis of Fourier coefficients.

6.2 Extracting Shifted Convolution Sums: Nonzero Shifts

Define $S_\xi(T)$ to be the contribution of the ξ th Fourier coefficient of ϕ to (48) as before. It remains to estimate $S_\xi(T)$ in terms of shifted convolution sums, which we do first when $\xi \neq 0$. Squaring out $|F_k(z)|^2$ and integrating in x gives

$$S_\xi(T) = 2^{-r_2} \sqrt{D} \sum_{\eta > 0} a_f(\eta) \overline{a_f(\eta + \xi)} \left(\int_{\mathbb{R}_+^r} g(Ty) a_\xi(y) \langle \mathbf{K}_k(\eta \kappa y), \mathbf{K}_k((\eta + \xi) \kappa y) \rangle Ny^{-1} dy^\times \right).$$

Applying Hölder's inequality, this becomes

$$S_\xi(T) \ll |a_\xi(T^{-1})| \sum_{\eta > 0} |a_f(\eta) a_f(\eta + \xi)| \left(\int_{\mathbb{R}_+^r} g(Ty) \left[\left(\frac{N(\eta + \xi)}{N\eta} \right)^{1/2} |\mathbf{K}_k(\eta \kappa y)|^2 + \left(\frac{N\eta}{N(\eta + \xi)} \right)^{1/2} |\mathbf{K}_k((\eta + \xi) \kappa y)|^2 \right] Ny^{-1} dy^\times \right).$$

The second term in this integral behaves identically to the first, and we ignore it for simplicity. Applying a change of variable and the normalisation $a_f(\eta) = \lambda_\pi(\eta) N \eta^{-1/2} a_f(1)$, we have

$$\begin{aligned} S_\xi(T) &\ll |a_\xi(T^{-1})| \sum_{\eta > 0} |a_f(\eta) a_f(\eta + \xi)| N(\eta(\eta + \xi))^{1/2} \\ &\quad \int_{\mathbb{R}_+^r} g(T|\eta \kappa|^{-1} y) |\mathbf{K}_k(y)|^2 Ny^{-1} dy^\times \\ &\ll |a_\xi(T^{-1})| |a_f(1)|^2 \sum_{\eta > 0} |\lambda_\pi(\eta) \lambda_\pi(\eta + \xi)| \\ &\quad \int_{\mathbb{R}_+^r} g(T|\eta \kappa|^{-1} y) |\mathbf{K}_k(y)|^2 Ny^{-1} dy^\times. \end{aligned} \quad (49)$$

We have the following formula for $|K_f|^2$ from (2) and (3),

$$|\mathbf{K}_k(y)|^2 = \prod_{i \leq r_1} y_i^{k_i} \exp(-4\pi y_i) \prod_{i > r_1} y_i^{k_i+2} \sum_{j=0}^{k_i} \binom{k_i}{j} |K_{k_i/2-j}(4\pi y_i)|^2,$$

where we now take $y_i \in \mathbb{R}$ for all i . For a multi-index $j = (j_i) \in J$ we define $K_{k,j}(y)$ to be the corresponding term in the formula for $\mathbf{K}_k(y)$ so that

$$|K_{k,j}(y)|^2 = \prod_{i \leq r_1} y_i^{k_i} \exp(-4\pi y_i) \prod_{i > r_1} y_i^{k_i+2} \binom{k_i}{j_i} |K_{k_i/2-j_i}(4\pi y_i)|^2, \quad (50)$$

and define $S_{\xi,j}(T)$ be the corresponding term in $S_{\xi}(T)$. We shall partition J as $J_0 \cup J_1$, where $J_0 = \{j \mid \min(j_i, k_i - j_i) > k_i^{1/2}\}$. The reason for separating the indices in this way is that for $j \in J_0$, the arguments of all the Bessel functions appearing in (50) are bounded away from $\pm k_i/2$. As a result, when we calculate the Mellin transforms of $|K_{k,j}(y)|^2$ the gamma factors which appear have arguments with large real parts, and so we may approximate them well using the Luo-Sarnak lemma. For $j \in J_0$ this lets us give good bounds for $S_{\xi,j}(T)$, while a weaker bound will suffice for the remaining terms because J_1 is small (in fact $|J_1| \ll \|k\|^{-\nu/2}|J|$). We begin by deriving this weak bound for all j , interchanging the sum and integral in (49) to obtain

$$S_{\xi,j}(T) \ll |a_{\xi}(T^{-1})||a_f(1)|^2 \int_{\mathbb{R}_+^r} |K_{k,j}(y)|^2 Ny^{-1} \sum_{\eta > 0} |\lambda_{\pi}(\eta)\lambda_{\pi}(\eta + \xi)| g(T|\eta\kappa|^{-1}y) dy^{\times}. \quad (51)$$

The inner function

$$\sum_{\eta > 0} |\lambda_{\pi}(\eta)\lambda_{\pi}(\eta + \xi)| g(T|\eta\kappa|^{-1}y)$$

is bounded above by the sum over η such that $Ty_i \ll |\eta_i| \ll Ty_i$ for all i , weighted by $|\lambda_{\pi}(\eta)\lambda_{\pi}(\eta + \xi)| \ll \|Ty\|^{\epsilon}$, from which it follows that

$$\sum_{\eta > 0} |\lambda_{\pi}(\eta)\lambda_{\pi}(\eta + \xi)| g(T|\eta\kappa|^{-1}y) \ll T^{n+\epsilon} Ny \prod_{i=1}^r (1 + y_i^{\epsilon}).$$

Applying this to (51) gives the upper bound

$$S_{\xi,j}(T) \ll T^{n+\epsilon} |a_{\xi}(T^{-1})||a_f(1)|^2 \int_{\mathbb{R}_+^r} |K_{k,j}(y)|^2 \prod_{i=1}^r (1 + y_i^{\epsilon}) dy^{\times}.$$

We may factorise this integral as a product over the Archimedean places, and at each place we will bound the product of the local integral and the corresponding terms of $|a_f(1)|^2$. The factor corresponding to a real place $i \leq r_1$ is

$$\begin{aligned} \frac{(4\pi)^{k_i}}{\Gamma(k_i)} \int_0^\infty y^{k_i} \exp(-4\pi y) (1+y^\epsilon) dy^\times &\ll \frac{1}{\Gamma(k_i)} (\Gamma(k_i) + \Gamma(k_i + \epsilon)) \\ &\ll k_i^\epsilon. \end{aligned}$$

For $i > r_1$, it is

$$\frac{(2\pi)^{k_i}}{\Gamma(k_i/2 + 1)^2} \binom{k_i}{j_i} \int_0^\infty y^{k_i+2} |K_{k_i/2-j_i}(4\pi y)|^2 (1+y^\epsilon) dy^\times,$$

and we may evaluate this using the following formula, taken from [10]:

$$\int_0^\infty y^\lambda K_\mu(y) K_\nu(y) dy = \frac{2^{\lambda-2}}{\Gamma(\lambda+1)} \prod_{\pm} \Gamma\left(\frac{1+\lambda \pm \mu \pm \nu}{2}\right). \quad (52)$$

Applying this, we obtain

$$\begin{aligned} \frac{1}{\Gamma(k_i/2 + 1)^2} \binom{k_i}{j_i} &\left(\frac{\Gamma(k_i/2 + 1)^2}{\Gamma(k_i + 2)} \Gamma(j_i + 1) \Gamma(k_i - j_i + 1) \right. \\ &\left. + \frac{\Gamma(k_i/2 + 1 + \epsilon/2)^2}{\Gamma(k_i + 2 + \epsilon)} \Gamma(j_i + 1 + \epsilon/2) \Gamma(k_i - j_i + 1 + \epsilon/2) \right) \ll k_i^{1-\epsilon}. \end{aligned}$$

Multiplying these local integrals gives the bound

$$S_{\xi,j}(T) \ll \frac{|a_\xi(T^{-1})|}{L(1, \text{sym}^2 \pi)} N k^\epsilon \prod_{i>r_1} k_i^{-1}, \quad (53)$$

and so the contribution to $S_\xi(T)$ from all $j \in J_1$ is bounded above by

$$\ll \frac{|a_\xi(T^{-1})|}{L(1, \text{sym}^2 \pi)} \|k\|^{-\nu/2+\epsilon}.$$

We shall treat the terms with $j \in J_0$ more carefully, by factorising the inner integral in (49) and using Mellin inversion to estimate each factor. As before, we shall pair each local integral with the corresponding factor from $|a_f(1)|^2$. For $i \leq r_1$ we need to consider

$$\frac{(4\pi)^{k_i}}{\Gamma(k_i)} \int_{\mathbb{R}^+} h(T|\eta_i \kappa_i|^{-1} y) y^{k_i-1} \exp(-4\pi y) dy^\times, \quad (54)$$

which by Mellin inversion is equal to

$$\frac{4\pi}{k_i - 1} \int_{(\sigma)} H(-s) \left(\frac{T}{4\pi|\eta_i \kappa_i|} \right)^s \frac{\Gamma(s + k_i - 1)}{\Gamma(k_i - 1)} ds. \quad (55)$$

Applying the Luo-Sarnak lemma to this, we have

$$(54) = \frac{4\pi}{k_i - 1} \int_{(\sigma)} H(-s) \left(\frac{T}{4\pi|\eta_i\kappa_i|} \right)^s (k_i - 1)^s (1 + O((|s| + 1)^2 k_i^{-1})) ds \\ \ll \frac{1}{k_i} \left(h \left(\frac{T(k_i - 1)}{4\pi|\eta_i\kappa_i|} \right) + \left| \int_{(\sigma)} H(-s) \left(\frac{T}{4\pi|\eta_i\kappa_i|} \right)^s (k_i - 1)^{s-1} (|s| + 1)^2 ds \right| \right).$$

Choosing $\sigma = 1 + \epsilon$ gives the final bound

$$(54) \ll \frac{1}{k_i} \left(h \left(\frac{T(k_i - 1)}{4\pi|\eta_i\kappa_i|} \right) + O(k_i^\epsilon (T/|\eta_i|)^{1+\epsilon}) \right). \quad (56)$$

For $i > r_1$, we must consider the integral

$$\frac{(2\pi)^{k_i}}{\Gamma(k_i/2 + 1)^2} \binom{k_i}{j_i} \int_{\mathbb{R}^+} h(T|\eta_i\kappa_i|^{-1}y) y^{k_i} |K_{k_i/2-j_i}(4\pi y)|^2 dy^\times. \quad (57)$$

Applying Mellin inversion using (52), this becomes

$$\ll \frac{1}{\Gamma(k_i/2 + 1)^2} \binom{k_i}{j_i} \int_{\sigma} H(-s) \left(\frac{T}{2\pi|\eta_i\kappa_i|} \right)^s \frac{\Gamma((s + k_i)/2)^2}{\Gamma(s + k_i)} \Gamma(s/2 + j_i) \Gamma(s/2 + k_i - j_i) ds.$$

Because $j \in J_0$, j_i and $k_i - j_i$ are $\geq k_i^{1/2}$ so we may apply (31) and choose $\sigma = 2 + \epsilon$ to obtain

$$(57) \ll \frac{1}{k_i(k_i - j_i)j_i} h \left(\frac{T\sqrt{j_i(k_i - j_i)}}{2\pi|\eta_i\kappa_i|} \right) + O(k_i^{-3/2+\epsilon} (T/|\eta_i|)^{2+\epsilon}). \quad (58)$$

Substituting the bounds (56) and (58), equation (49) becomes

$$S_{\xi,j}(T) \ll \frac{|a_\xi(T^{-1})|}{L(1, \text{sym}^2\pi)} \sum_{\eta > 0} |\lambda_\pi(\eta)\lambda_\pi(\eta + \xi)| \prod_{i \leq r_1} \frac{1}{k_i} \left(h \left(\frac{T(k_i - 1)}{4\pi|\eta_i\kappa_i|} \right) + O(k_i^\epsilon (T/|\eta_i|)^{1+\epsilon}) \right) \\ \times \prod_{i > r_1} \left(\frac{1}{k_i(k_i - j_i)j_i} h \left(\frac{T\sqrt{j_i(k_i - j_i)}}{2\pi|\eta_i\kappa_i|} \right) + O(k_i^{-3/2+\epsilon} (T/|\eta_i|)^{2+\epsilon}) \right).$$

As in the totally real case we may use the bound $|\lambda_\pi(\eta)| \ll N\eta^\epsilon$ to show that the contribution to the sum from all error terms is $O(|J|^{-1} \|k\|^{-\nu/2+\epsilon} T^{n+\epsilon})$, so our upper bound may be rewritten

$$S_{\xi,j}(T) \ll \frac{|a_\xi(T^{-1})|}{L(1, \text{sym}^2\pi)} \left(\sum_{\eta > 0} |\lambda_\pi(\eta)\lambda_\pi(\eta + \xi)| \prod_{i \leq r_1} \frac{1}{k_i} h \left(\frac{T(k_i - 1)}{4\pi|\eta_i\kappa_i|} \right) \right. \\ \left. \times \prod_{i > r_1} \frac{1}{k_i(k_i - j_i)j_i} h \left(\frac{T\sqrt{j_i(k_i - j_i)}}{2\pi|\eta_i\kappa_i|} \right) + O(|J|^{-1} \|k\|^{-\nu/2+\epsilon} T^{n+\epsilon}) \right).$$

On summing over j it can be seen that we have proven the inequalities (40) and (41), where the terms for $j \in J_1$ are absorbed into the error term.

6.3 Extracting Shifted Convolution Sums: The Zero Shift

Having dealt with the ‘error’ terms with $\xi \neq 0$, it remains to prove (38) and (39) by considering the ‘main’ term $S_0(T)$, which by (43) reduces to studying the integral $I_1(T)$ as in the totally real case. Squaring out $|F_k(z)|^2$ and integrating in x , we obtain

$$I_1(T) = 2^{-r_2} \sqrt{|D|} \sum_{\eta > 0} |a_f(\eta)|^2 \int_{\mathbb{R}_+^r} g(Ty) |\mathbf{K}_k(\eta \kappa y)|^2 N y^{-1} dy^\times.$$

Applying the normalisation of $a_f(\eta)$, we have

$$I_1(T) = 2^{-r_2} \sqrt{|D|} |a_f(1)|^2 N \kappa \sum_{\eta > 0} |\lambda_\pi(\eta)|^2 \int_{\mathbb{R}_+^r} g(Ty) N(\eta \kappa y)^{-1} |\mathbf{K}_k(\eta \kappa y)|^2 dy^\times.$$

As with the non-zero shifts, we may expand this into a sum over the multi-indices $j \in J$, and denote the j th term by $I_{1,j}(T)$. If we define the symmetrised functions \tilde{g} and $\tilde{\psi}_j$ by

$$\tilde{g}(y) = \sum_{u \in \mathcal{O}_\times^+} g(uy), \quad \tilde{\psi}_j(y) = N y^{-1} \sum_{u \in \mathcal{O}_\times^+} |K_{k,j}(uy)|^2,$$

then $I_{1,j}(T)$ may be expressed as

$$\begin{aligned} I_{1,j}(T) &= 2^{-r_2} \omega_+ \sqrt{|D|} |a_f(1)|^2 N \kappa \sum_{(\eta) > 0} |\lambda_\pi(\eta)|^2 \int_{\mathbb{R}_+^r} g(Ty) \tilde{\psi}_j(\eta \kappa y) dy^\times \\ &= 2^{-r_2} \omega_+ \sqrt{|D|} |a_f(1)|^2 N \kappa \sum_{(\eta) > 0} |\lambda_\pi(\eta)|^2 \int_{\mathbb{R}_+^r / \mathcal{O}_\times^+} \tilde{g}(Ty) \tilde{\psi}_j(\eta \kappa y) dy^\times. \end{aligned}$$

Note that the factor of ω_+ arises because the quotient of \mathcal{O}^+ by \mathcal{O}_+^\times contains each ideal with this multiplicity. If we let $G(s, m)$ be the Mellin transform of \tilde{g} as in (22) then Mellin inversion gives

$$\begin{aligned} \int_{\mathbb{R}_+^r / \mathcal{O}_\times^+} \tilde{g}(Ty) \tilde{\psi}_j(\eta \kappa y) dy^\times &= \\ &= \frac{1}{2\pi i V_c} \sum_m \int_{(\sigma)} G(-s, -m) \left(\frac{T}{4\pi}\right)^{ns} N(\eta \kappa)^{-s} \lambda_{-m}(\eta \kappa) \Gamma(k, j, s, m) ds, \end{aligned}$$

where $\Gamma(k, j, s, m)$ is the Mellin transform of $\tilde{\psi}_j$ and is given by

$$\begin{aligned}\Gamma(k, j, s, m) &= \prod_{i \leq r_1} (4\pi)^{-k_i+1} \Gamma(s + \beta(m, i) + k_i - 1) \\ &\quad \times \prod_{i > r_1} (2\pi)^{-k_i} \binom{k_i}{j_i} 2^{2s+\beta(m, i)} \frac{\Gamma(s + (\beta(m, i) + k_i)/2)^2}{8\Gamma(2s + \beta(m, i) + k_i)} \\ &\quad \Gamma(s + \beta(m, i)/2 + j_i) \Gamma(s + \beta(m, i)/2 + k_i - j_i).\end{aligned}$$

Substituting into $I_{1,j}(T)$ and forming the L -function from the sum over η , we have

$$\begin{aligned}I_{1,j}(T) &= \frac{\omega_+ \sqrt{|D|}}{2r_2 V_c} |a_f(1)|^2 \frac{1}{2\pi i} \sum_m \int_{(\sigma)} G(-s, -m) \left(\frac{T}{4\pi}\right)^{ns} N\kappa^{1-s} \lambda_{-m}(\kappa) \\ &\quad L(s, \text{sym}^2 \pi \otimes \lambda_{-m}) \frac{L(s, \lambda_{-m})}{L(2s, \lambda_{-2m})} \Gamma(k, j, s, m) ds.\end{aligned}$$

We now substitute the value of $|a_f(1)|^2$ and shift the line of integration to $\sigma = 1/2$, giving

$$I_{1,j}(T) = \frac{C_g T^n}{|J|} + E_{1/2,j}(T) \quad (59)$$

with

$$\begin{aligned}E_{1/2,j}(T) &\ll \frac{1}{L(1, \text{sym}^2 \pi)} \prod_{i \leq r_1} \frac{(4\pi)^{k_i}}{\Gamma(k_i)} \prod_{i > r_1} \frac{(2\pi)^{k_i}}{\Gamma(k_i/2 + 1)^2} \binom{k_i}{j_i} \sum_m \int_{(1/2)} G(-s, -m) \left(\frac{T}{4\pi}\right)^{ns} \\ &\quad L(s, \text{sym}^2 \pi \otimes \lambda_{-m}) \frac{L(s, \lambda_{-m})}{L(2s, \lambda_{-2m})} \Gamma(k, j, s, m) ds.\end{aligned}$$

By Stirling's formula and the rapid decay of $G(s, m)$ this error may be bounded above by

$$\begin{aligned}E_{1/2,j}(T) &\ll \left(\frac{T^n}{Nk}\right)^{1/2} \frac{1}{L(1, \text{sym}^2 \pi)} \prod_{i > r_1} \frac{1}{\sqrt{j_i(k_i - j_i)}} \\ &\quad \sum_m \int_{-\infty}^{+\infty} \frac{|L(1/2 + it, \text{sym}^2 \pi \otimes \lambda_{-m})|}{(|t| + \|m\| + 1)^A} dt \quad (60)\end{aligned}$$

for any $A > 0$. Because $x^{1/2}$ is integrable at 0,

$$\sum_{j \in J} \prod_{i > r_1} \frac{1}{\sqrt{j_i(k_i - j_i)}}$$

is bounded independently of k so that when we sum (59) and (60) over j we obtain

$$I_1(T) = cT^n + O(T^{n/2}R_k(f)),$$

with

$$R_k(f) = \frac{1}{\sqrt{Nk}L(1, \text{sym}^2\pi)} \sum_m \int_{-\infty}^{+\infty} \frac{|L(1/2 + it, \text{sym}^2\pi \otimes \lambda_{-m})|}{(|t| + \|m\| + 1)^A} dt.$$

This completes the proof of proposition 13.

7 Application of the Large Sieve

In this section we complete the proof of theorem 4 by establishing the bounds of proposition 10 for the shifted sums

$$C_\xi(x) = \sum_{\eta \leq x} |\lambda_1(\eta)\lambda_2(\eta + \xi)|,$$

where λ_i are multiplicative functions on \mathcal{O}^+ satisfying $|\lambda_i(\eta)| \leq \tau_m(\eta)$ for some m and $x = (x_i)$ satisfies $x_i \geq \|x\|^\nu$ for some $\nu > 0$. We first rearrange and partition the sums into pieces which may be treated either by elementary methods or by a large sieve. We assume that $0 < \|\xi\| \leq \|x\|^\nu$, and given $\epsilon > 0$ we will be working throughout with a choice of variables satisfying

$$z = \|x\|^{1/s} \quad \text{with } s = \epsilon \log \log x, \tag{61}$$

$$y = \|x\|^\epsilon. \tag{62}$$

We factorise the ideals (η) and $(\eta + \xi)$ as

$$(\eta) = \mathfrak{a}\mathfrak{b} \quad \text{and} \quad (\eta + \xi) = \mathfrak{a}_\xi \mathfrak{b}_\xi$$

in such a way that for every prime ideal \mathfrak{p} dividing $\eta(\eta + \xi)$,

$$\mathfrak{p}|\mathfrak{a}\mathfrak{a}_\xi \Rightarrow N\mathfrak{p} \leq z \quad \text{and} \quad \mathfrak{p}|\mathfrak{b}\mathfrak{b}_\xi \Rightarrow N\mathfrak{p} > z,$$

and partition the sum $C_\xi(x)$ into parts depending on the norm of \mathfrak{a} and \mathfrak{a}_ξ . We denote by $C^y(x)$ the part of $C_\xi(x)$ in which either $N\mathfrak{a}$ or $N\mathfrak{a}_\xi$ is greater than y ,

$$C^y(x) = \sum_{\substack{\eta \leq x \\ N\mathfrak{a} > y}} |\lambda_1(\eta)\lambda_2(\eta + \xi)| + \sum_{\substack{\eta \leq x \\ N\mathfrak{a}_\xi > y}} |\lambda_1(\eta)\lambda_2(\eta + \xi)|,$$

and the part where both $N\mathfrak{a}$ and $N\mathfrak{a}_\xi$ are less than or equal to y we denote by $C_y(x)$,

$$C_y(x) = \sum_{\substack{\eta \leq x \\ N\mathfrak{a}, N\mathfrak{a}_\xi \leq y}} |\lambda_1(\eta)\lambda_2(\eta + \xi)|,$$

so that $C_\xi(x) = C^y(x) + C_y(x)$.

7.1 Treating $C^y(x)$ by Elementary Methods

We first handle the terms with $N\mathfrak{a}$ or $N\mathfrak{a}_\xi$ large. We begin by applying Hölder's inequality and $|\lambda_i(\eta)| \leq \tau_m(\eta)$ to get

$$C^y(x) \ll \left(\sum_{\substack{\eta \leq x \\ N\mathfrak{a} > y}} 1 \right)^{1/2} \left(\sum_{\eta \leq x + \|\xi\|} \tau_m^4(\eta) \right)^{1/2}.$$

We know that $x + \|\xi\| \leq 2x$ by our assumption on ξ , and have the bound

$$\sum_{\eta \leq 2x} \tau_m^4(\eta) \ll Nx(\log \|x\|)^A$$

for some A . As all prime factors of $N\mathfrak{a}$ must be at most z , we may use a Rankin's method argument ([25], Thm. 7.6) to bound the number of allowable values of $N\mathfrak{a}$ up to t by $t(\log t)^{-A}$ for all A . Combined with a bound of $\ll (\log t)^n$ for the number of \mathfrak{a} with norm t , we see that the number of choices for \mathfrak{a} with $N\mathfrak{a} \leq t$ is $\ll t(\log t)^{-A}$. Partial summation and our choice of x, y and z then gives the bound

$$\sum_{\substack{(\eta): N\eta \leq Nx \\ N\mathfrak{a} > y}} 1 \ll \frac{Nx}{(\log \|x\|)^A}$$

for any A , and the upper bound of $(\log \|x\|)^{n-1}$ for the number of $\eta \leq x$ generating a given (η) lets us conclude

$$C^y(x) \leq \frac{Nx}{(\log \|x\|)^2}. \tag{63}$$

7.2 Treating $C_y(x)$ by the Large Sieve

From our definition of $C_y(x)$, we are left with evaluating

$$C_y(x) \ll \sum_{\substack{N\mathfrak{a}, N\mathfrak{a}_\xi \leq y \\ \mathfrak{p}|\mathfrak{a}\mathfrak{a}_\xi \Rightarrow N\mathfrak{p} \leq z}} |\lambda_1(\mathfrak{a})\lambda_2(\mathfrak{a}_\xi)| \sum_{\substack{\eta \leq x \\ \eta \equiv 0 \pmod{\mathfrak{a}} \\ \eta \equiv -\xi \pmod{\mathfrak{a}_\xi} \\ \mathfrak{p}|\mathfrak{b}\mathfrak{b}_\xi \Rightarrow N\mathfrak{p} > z}} |\lambda_1(\mathfrak{b})\lambda_2(\mathfrak{b}_\xi)|.$$

To help deal with certain co-primality conditions which come up during our analysis, we pull out the greatest common divisor \mathfrak{v} of \mathfrak{a} and \mathfrak{a}_ξ , which we choose to have a normalised positive generator v . Writing $\eta_v = \eta/v$ and $\eta_v + w = (\eta + \xi)/v$, we again factorise (η_v) and $(\eta_v + w)$ with $(\mathfrak{a}, \mathfrak{a}_\xi) = (\mathfrak{a}\mathfrak{a}_\xi, w) = \mathcal{O}$, so that

$$C_y(x) \ll \sum_{\substack{vw=\xi \\ v \text{ normalised}}} \sum_{\substack{N\mathfrak{a}, N\mathfrak{a}_\xi \leq y/N\mathfrak{v} \\ \mathfrak{p}|\mathfrak{a}\mathfrak{a}_\xi \Rightarrow N\mathfrak{p} \leq z \\ (\mathfrak{a}, \mathfrak{a}_\xi) = (\mathfrak{a}\mathfrak{a}_\xi, w) = \mathcal{O}}} |\lambda_1(\mathfrak{v}\mathfrak{a})\lambda_2(\mathfrak{v}\mathfrak{a}_\xi)| \sum_{\substack{\eta_v \ll x/\|v\| \\ \eta_v \equiv r \pmod{\mathfrak{a}\mathfrak{a}_\xi} \\ \mathfrak{p}|\mathfrak{b}\mathfrak{b}_\xi \Rightarrow N\mathfrak{p} > z}} |\lambda_1(\mathfrak{b})\lambda_2(\mathfrak{b}_\xi)|. \quad (64)$$

Here we applied the Chinese remainder theorem so that the residue class r in the innermost sum satisfies $r \equiv 0 \pmod{a}$ and $r \equiv -w \pmod{a_\xi}$. The Ramanujan-Petersson conjecture and our choice of s in (61) imply that $|\lambda_1(\mathfrak{b})\lambda_2(\mathfrak{b}_\xi)| \ll (\log \|x\|)^{2m\epsilon}$, as we have $|\lambda_1(\mathfrak{p}^\alpha)| \leq \tau_m(\mathfrak{p}^\alpha) \leq 2^{\alpha+m-1}$ and $\mathfrak{b} = \mathfrak{p}_1^{\alpha_1} \dots \mathfrak{p}_t^{\alpha_t}$ with $\alpha_1 + \dots + \alpha_t \leq s$. We may therefore substitute this, and proceed to bound the count

$$\sum_{\substack{\eta_v \ll x/\|v\| \\ \eta_v \equiv r \pmod{\mathfrak{a}\mathfrak{a}_\xi} \\ \mathfrak{p}|\mathfrak{b}\mathfrak{b}_\xi \Rightarrow N\mathfrak{p} > z}} 1. \quad (65)$$

Choose normalised generators a and a_ξ for \mathfrak{a} and \mathfrak{a}_ξ , which will satisfy $\|a\|, \|a_\xi\| \ll y^{1/n}/\|v\|$. Writing $\eta_v = aa_\xi m + r$ with r chosen in a negative fundamental domain for $\mathbb{F}/(aa_\xi)$ (so that $aa_\xi m \geq 0$), we note the following equivalences between divisibility conditions for primes with $N\mathfrak{p} \leq z$:

$$\begin{aligned} \mathfrak{p} \nmid b &\iff \mathfrak{p} \nmid (a_\xi m + r/a), \\ \mathfrak{p} \nmid b_\xi &\iff \mathfrak{p} \nmid (am + (r+w)/a_\xi). \end{aligned}$$

For fixed normalised a and a_ξ satisfying $(a, a_\xi) = (aa_\xi, w) = \mathcal{O}$ and $\|a\|, \|a_\xi\| \ll y^{1/n}/\|v\|$, we see that the count in (65) is bounded by $S = |\mathcal{S}(\mathcal{M}, \mathcal{P}, \Omega)|$ where we define $\mathcal{S}(\mathcal{M}, \mathcal{P}, \Omega)$ to be the ‘sifted set,’

$$\mathcal{S}(\mathcal{M}, \mathcal{P}, \Omega) = \{m \in \mathcal{M} \mid m \pmod{\mathfrak{p}} \notin \Omega_{\mathfrak{p}} \text{ for all } \mathfrak{p} \in \mathcal{P}\}.$$

Here,

$$\begin{aligned} \mathcal{M} &= \{m \in \mathcal{O} \mid 0 < mvaa_\xi \ll x\}, \\ \mathcal{P} &= \{\mathfrak{p} \mid 2 < N\mathfrak{p} \leq z\}, \end{aligned}$$

and the set $\Omega = \bigcup_{\mathfrak{p} \in \mathcal{P}} \Omega_{\mathfrak{p}}$ of residue classes to be ‘sieved out’ is given by

$$\Omega_{\mathfrak{p}} = \begin{cases} \{r_1 \pmod{\mathfrak{p}}\} & \text{for } \mathfrak{p} \mid a \\ \{r_2 \pmod{\mathfrak{p}}\} & \text{for } \mathfrak{p} \mid a_\xi \\ \{r_1, r_2 \pmod{\mathfrak{p}}\} & \text{for } \mathfrak{p} \nmid aa_\xi, \end{cases}$$

where $r_1 \equiv -\overline{a_\xi}r/a(\mathfrak{p})$ and $r_2 \equiv -\overline{a}(r+w)/a_\xi(\mathfrak{p})$. Here the overline means multiplicative inverse mod \mathfrak{p} .

We now apply a variant of the standard large sieve for the lattice \mathbb{Z}^n . Let $d = (d_i)$ with $d_i > \|d\|^\nu$ for some $\nu > 0$, let $B(d)$ be the box with dimensions d centred at the origin in \mathbb{R}^n , and $D(d)$ be the image of $B(d)$ under any rotation. If \mathcal{P} is a set of rational primes, define Ω_p to be a subset of L/pL of cardinality $\omega(p)$ for each $p \in \mathcal{P}$, and define a sifted set $\mathcal{S}(\mathcal{L}, \mathcal{P}, \Omega)$ by

$$\begin{aligned} \mathcal{S}(\mathcal{L}, \mathcal{P}, \Omega) &= \{m \in \mathcal{L}; m \pmod{p} \notin \Omega_p \text{ for all } p \in \mathcal{P}\}, \\ \text{with } \mathcal{L} &= \mathbb{Z}^n \cap D(d). \end{aligned}$$

We then have

$$|\mathcal{S}(\mathcal{L}, \mathcal{P}, \Omega)| \ll_\nu \frac{Nd + Q^{2n}}{H}$$

for any $\|d\|^{\nu/2} \geq Q \geq 1$, where

$$H = \sum_{q \leq Q} h(q)$$

and $h(q)$ is the multiplicative function supported on squarefree integers with prime divisors in \mathcal{P} such that

$$h(p) = \frac{\omega(p)}{p^n - \omega(p)}.$$

This form of the large sieve may be proven using soft techniques of Poisson summation, described in chapter 7 of [17]. To apply this in the number field, identify \mathcal{O} with \mathbb{Z}^n and for each p , construct a set Ω_p from the $\Omega_{\mathfrak{p}}$ with $\mathfrak{p}|p$ using the Chinese remainder theorem. We then have

$$\omega(p) \geq (\alpha_p + \beta_p)p^{n-1} + O(p^{n-2}),$$

where α_p is the number of degree 1 primes above p and β_p is the number which do not divide aa_ξ . We then have the lower bound

$$H \gg (\log z)^2 \prod_{\mathfrak{p}|aa_\xi} \left(1 - \frac{1}{N\mathfrak{p}}\right),$$

so that the count (65) is bounded by

$$\ll \frac{Nx}{(\log z)^2 N(vaa_\xi)} \prod_{\mathfrak{p}|aa_\xi} \left(1 - \frac{1}{N\mathfrak{p}}\right)^{-1}.$$

Plugging this back into (64), we obtain

$$C^y(x) \ll \frac{(\log \|x\|)^{2m\epsilon} Nx}{(\log z)^2} \sum_{\substack{vw=\xi \\ v \text{ normalised}}} \sum_{\substack{Na, Na_\xi \leq y/Nv \\ \mathfrak{p} | aa_\xi \Rightarrow N\mathfrak{p} \leq z \\ (a, a_\xi) = (aa_\xi, w) = \mathcal{O}}} \frac{|\lambda_1(\mathfrak{v}a)\lambda_2(\mathfrak{v}a_\xi)|}{N(vaa_\xi)} \prod_{\mathfrak{p} | aa_\xi} \left(1 - \frac{1}{N\mathfrak{p}}\right)^{-1}.$$

For each v , we may bound the inner sum from above by an Euler product. If $\mathfrak{p} \nmid v$, the corresponding term is

$$1 + \frac{|\lambda_1(\mathfrak{p})| + |\lambda_2(\mathfrak{p})|}{N\mathfrak{p}} + O(N\mathfrak{p}^{-2+\epsilon}) \quad (66)$$

by our bounds on $|\lambda_i(\mathfrak{p})|$, and if $\mathfrak{p} | v$ it is

$$\frac{|\lambda_1(\mathfrak{p})\lambda_2(\mathfrak{p})|}{N\mathfrak{p}} + O(N\mathfrak{p}^{-2+\epsilon}). \quad (67)$$

(67) is at most 1 for almost all \mathfrak{p} , and so for any v we may bound the inner sum by

$$\ll \prod_{N\mathfrak{p} \leq z} \left(1 + \frac{|\lambda_1(\mathfrak{p})| + |\lambda_2(\mathfrak{p})|}{N\mathfrak{p}}\right).$$

This gives the bound

$$C^y(x) \ll \frac{\tau(\xi)Nx}{(\log \|x\|)^{2-\epsilon}} \prod_{N\mathfrak{p} \leq z} \left(1 + \frac{|\lambda_1(\mathfrak{p})| + |\lambda_2(\mathfrak{p})|}{N\mathfrak{p}}\right)$$

for $C^y(x)$, and when combined with our partition of $C_\xi(x)$ and the bound (63) this concludes the proof of proposition 10.

8 Proof of Theorem 5

In this section we shall prove theorem 5 by extending Soundararajan's approach of weak subconvexity to a number field. We prove the necessary triple product identities in section 8.1, before showing that the triple product L functions which appear satisfy the hypotheses of Soundararajan's theorem in section 8.2.

8.1 Triple Products

Throughout this section, C will denote a constant depending only on F which may vary from equation to equation. We shall also let σ denote the conjugate linear automorphism of π corresponding to complex conjugation on X , which has the property that $\langle \sigma(u), \sigma(v) \rangle = \overline{\langle u, v \rangle}$. We begin with the following triple product identity in the case of ϕ a Hecke-Maass cusp form.

Proposition 14. *Let ϕ be a Hecke-Maass cusp form with associated automorphic representation π' . Then*

$$|\langle \phi F_k, F_k \rangle|^2 = C \frac{\Lambda(\frac{1}{2}, \pi \otimes \pi \otimes \pi')}{\Lambda(1, \text{sym}^2 \pi)^2 \Lambda(1, \text{sym}^2 \pi')} \quad (68)$$

$$\sim_{\phi} N k^{-1} \frac{L(\frac{1}{2}, \text{sym}^2 \pi \otimes \pi')}{L(1, \text{sym}^2 \pi)^2} \quad (69)$$

where \sim_{ϕ} means that the ratio of the two quantities is bounded between two positive constants depending only on ϕ .

Proof. Because $|F_k|^2 dv$ is the pushforward of $|R_{\pi}(v_k)|^2 dx$, the inner product $\langle \phi F_k, F_k \rangle$ is equal to

$$\int_X |R_{\pi}(v_k)|^2 \phi dx = \int_X R_{\pi}(v_k) R_{\pi}(\sigma(v_k)) \phi dx,$$

and we may evaluate the RHS of this expression using Ichino's formula. Let $I = \otimes I_i$ and $I' = \otimes I'_i$ be the products of the Archimedean local factors of π and π' , k_i and r'_i be the relevant parameters of these local factors, and $u \in I'$ be the unit spherical vector. As all our vectors are unramified and our division algebra is split, the statement of Ichino's formula in this case is

$$\left| \int_X R_{\pi}(v_k) R_{\pi}(\sigma(v_k)) \phi dx \right|^2 = C \prod_{i=1}^r \int_{G_i} \langle I_i(g) v_{k_i}, v_{k_i} \rangle \langle I_i(g) v_{-k_i}, v_{-k_i} \rangle \langle I'_i(g) u_i, u_i \rangle d\overline{g_i} \frac{L(\frac{1}{2}, \pi \otimes \pi \otimes \pi')}{L(1, \text{sym}^2 \pi)^2 L(1, \text{sym}^2 \pi')}, \quad (70)$$

where $v_k = \otimes v_{k_i}$ and $u = \otimes u_i$. If ν_i is a complex place the i th local integral appearing in the product was computed in [23] to be

$$C \frac{\Gamma\left(\frac{1+k_i \pm i r'_i}{2}\right)^2 \Gamma\left(\frac{1 \pm i r'_i}{2}\right)^2}{\Gamma(1 + \frac{k_i}{2})^4 \Gamma(1 \pm i r'_i)^2}, \quad (71)$$

and up to an absolute constant this is equal to the ratio of the Archimedean factors at the place ν_i of the L functions appearing in (70). In the real case the local integral may be determined by comparison with Watson's formula, and is

$$C \frac{\Gamma_{\mathbb{R}}(k_i - 1/2 \pm i r'_i) \Gamma_{\mathbb{R}}(k_i + 1/2 \pm i r'_i) \Gamma_{\mathbb{R}}(1/2 \pm i r'_i) \Gamma_{\mathbb{R}}(3/2 \pm i r'_i)}{\Gamma_{\mathbb{R}}(k_i - 1/2)^2 \Gamma_{\mathbb{R}}(k_i + 1/2)^2 \Gamma_{\mathbb{R}}(1/2 \pm 2i r'_i)}. \quad (72)$$

This is again proportional to the relevant Archimedean factors of the L functions appearing in (70), which gives (68). Finally, the Archimedean factors (72) and (71) have the asymptotic behaviours k_i^{-1} and k_i^{-2} as $k_i \rightarrow \infty$, which gives (69). □

We now treat the inner products $\langle E(s, m, \cdot)F_k, F_k \rangle$ against spherical Eisenstein series by unfolding, to obtain the following formula.

Proposition 15.

$$|\langle E(s, m, \cdot)F_k, F_k \rangle| = C \left| \frac{\Lambda(1/2 + it, \pi \otimes \pi \otimes \lambda_{-m})}{\Lambda(1, \text{sym}^2 \pi) \Lambda(1 + 2it, \lambda_{-2m})} \right| \quad (73)$$

$$|\langle E(s, m, \cdot)F_k, F_k \rangle| \ll \frac{(1 + |t| + \|m\|)^{n/4+\epsilon}}{Nk} \left| \frac{L(1/2 + it, \text{sym}^2 \pi \otimes \lambda_{-m})}{L(1, \text{sym}^2 \pi)} \right|. \quad (74)$$

Proof. For $\text{Re}(s) > 1$, by unfolding and substituting the Fourier expansion of F_k we have

$$\begin{aligned} \langle E(s, m, \cdot)F_k, F_k \rangle &= \int_{\Gamma_\infty \backslash \mathbb{H}_F} Ny^s \lambda_m(y) |F_k(z)|^2 dv \\ &= |a_f(1)|^2 \int_{\Gamma_\infty \backslash \mathbb{H}_F} Ny^s \lambda_m(y) \sum_{\xi \in \mathcal{O}} |\lambda_\pi(\xi)|^2 N\xi^{-1} |\mathbf{K}_k(\xi \kappa y)|^2 dv. \end{aligned}$$

$\Gamma_\infty \backslash \mathbb{H}_F \simeq \mathbb{F}/\mathcal{O}\mu_+ \times \mathbb{R}_+^r/\mathcal{O}_+^\times$, where μ_+ acts on \mathbb{F}/\mathcal{O} by multiplication, and the volume of $\mathbb{F}/\mathcal{O}\mu_+$ is $2^{-r_2}\omega_+^{-1}\sqrt{|D|}$. We therefore have

$$\begin{aligned} \langle E(s, m, \cdot)F_k, F_k \rangle &= |a_f(1)|^2 2^{-r_2}\omega_+^{-1}\sqrt{|D|} \int_{\mathbb{R}_+^r/\mathcal{O}_+^\times} Ny^s \lambda_m(y) \\ &\quad \sum_{\xi \in \mathcal{O}} |\lambda_\pi(\xi)|^2 N\xi^{-1} |\mathbf{K}_k(\xi \kappa y)|^2 Ny^{-1} dy^\times. \end{aligned}$$

Making the change of variable $y \mapsto |\xi \kappa|^{-1}y$ and unfolding the integral over \mathcal{O}_+^\times , this becomes

$$\begin{aligned} &= |a_f(1)|^2 2^{-r_2}\sqrt{|D|} N\kappa \sum_{(\xi)} |\lambda_\pi(\xi)|^2 (N\xi \kappa)^{-s} \lambda_{-m}(\xi \kappa) \\ &\quad \int_{\mathbb{R}_+^r} Ny^s \lambda_m(y) |\mathbf{K}_k(y)|^2 Ny^{-1} dy^\times \\ &= |a_f(1)|^2 2^{-r_2}\sqrt{|D|} N\kappa^{1-s} \lambda_{-m}(\kappa) \frac{L(s, \pi \otimes \pi \otimes \lambda_{-m})}{L(2s, \lambda_{-2m})} \\ &\quad \int_{\mathbb{R}_+^r} Ny^s \lambda_m(y) |\mathbf{K}_k(y)|^2 Ny^{-1} dy^\times. \end{aligned}$$

(Note that the factor of ω_+^{-1} vanished because $\mathcal{O}/\mathcal{O}_+^\times$ counts every ideal with this multiplicity.) We factorise the integral occurring here, and pair each factor with the corresponding term of $|a_f(1)|^2$ so that

$$\langle E(1/2 + it, m, \cdot) F_k, F_k \rangle = C \frac{L(1/2 + it, \pi \otimes \pi \otimes \lambda_{-m})}{L(1, \text{sym}^2 \pi) L(1 + 2it, \lambda_{-2m})} \prod_{i=1}^r \mathcal{T}_i, \quad (75)$$

where for $i \leq r_1$ we have

$$\mathcal{T}_i = \frac{(4\pi)^{k_i}}{\Gamma(k_i)} \int_0^\infty y^{1/2+it+\beta(m,i)} |\mathbf{K}_i(y)|^2 y^{-1} dy^\times,$$

and for $i > r_1$

$$\mathcal{T}_i = \frac{(2\pi)^{k_i}}{\Gamma(k_i/2 + 1)^2} \int_0^\infty y^{1+2it+\beta(m,i)} |\mathbf{K}_i(y)|^2 y^{-2} dy^\times.$$

The integral at real places may be easily calculated to be

$$\mathcal{T}_i = (4\pi)^{1/2-it-\beta(m,i)} \frac{\Gamma(k_i - 1/2 + it + \beta(m,i))}{\Gamma(k_i)},$$

and the integral at complex places was calculated in [23] to have absolute value

$$\mathcal{T}_i = \frac{\Gamma(1/2 \pm (it + \beta(m,i)) + k_i/2) \Gamma(1/2 \pm (it + \beta(m,i)))}{\Gamma(1 + \frac{k_i}{2})^2 |\Gamma(1 + 2it + 2\beta(m,i))|}.$$

Both of these terms agree in absolute value with the ratio of gamma factors at the corresponding infinite place of the L functions appearing in (75), which proves formula (73). To prove (74), we use Stirling together with the bound $|\Gamma(\sigma + it)| \leq \Gamma(\sigma)$ to show that $|\mathcal{T}_i| \ll k_i^{-1/2}$ for ν_i real and $\mathcal{T}_i \ll k_i^{-1}(1 + |t + \beta(m,i)|)^{-1/2} \leq k_i^{-1}$ for ν_i complex. This gives

$$\begin{aligned} |\langle E(1/2 + it, m, \cdot) F_k, F_k \rangle| &\ll \left| N k^{-1/2} \frac{L(1/2 + it, \pi \otimes \pi \otimes \lambda_{-m})}{L(1, \text{sym}^2 \pi) L(1 + 2it, \lambda_{-2m})} \right| \\ &= N k^{-1/2} \left| \frac{L(1/2 + it, \text{sym}^2 \pi \otimes \lambda_{-m}) L(1/2 + it, \lambda_{-m})}{L(1, \text{sym}^2 \pi) L(1 + 2it, \lambda_{-2m})} \right|, \end{aligned}$$

and applying the convex bound $L(1/2 + it, \lambda_{-m}) \ll (1 + |t| + \|m\|)^{n/4+\epsilon}$ and the lower bound $L(1 + 2it, \lambda_{-2m}) \gg (1 + |t| + \|m\|)^{-\epsilon}$ yields (74). \square

8.2 Weak Subconvexity

Having expressed the inner products $\langle \phi F_k, F_k \rangle$ for ϕ a Hecke-Maass cusp form or Eisenstein series in terms of L values, we now prove theorem 5 by applying the weak subconvexity of Soundararajan [36] to these values. This is a theorem which is valid for any Dirichlet series $L(s, \pi)$ over the rationals satisfying certain conditions, which we now describe. The first of these is that $L(s, \pi)$ may be given by an Euler product

$$L(s, \pi) = \sum_{n=1}^{\infty} \frac{a_{\pi}(n)}{n^s} = \prod_p \prod_{j=1}^m \left(1 - \frac{\alpha_{j,\pi}(p)}{p^s} \right)^{-1},$$

and that both the series and product are absolutely convergent for $\operatorname{Re}(s) > 1$ (the notation $L(s, \pi)$ is meant to suggest that π corresponds to an automorphic representation, although this is not assumed). The second is that there is an Archimedean component

$$L_{\infty}(s, \pi) = N^{s/2} \prod_{j=1}^m \Gamma_{\mathbb{R}}(s + \mu_j)$$

for $N \in \mathbb{Z}$ and $\mu_j \in \mathbb{C}$, such that the completed L function $\Lambda(s, \pi) = L_{\infty}(s, \pi)L(s, \pi)$ has an analytic continuation to the entire complex plane. Moreover, it should satisfy a functional equation

$$\Lambda(s, \pi) = \kappa \Lambda(1 - s, \tilde{\pi}),$$

for κ a complex number of absolute value one and where

$$L(s, \tilde{\pi}) = \sum_{n=1}^{\infty} \frac{\overline{a_{\pi}(n)}}{n^s}, \quad \text{and} \quad L_{\infty}(s, \tilde{\pi}) = N^{s/2} \prod_{j=1}^m \Gamma_{\mathbb{R}}(s + \overline{\mu_j}).$$

These conditions are quite general, and hold for all the L functions appearing in our triple product identities. In addition, we require some bounds towards the Ramanujan-Selberg conjectures for π , which predicts that $|\alpha_{j,\pi}(p)| \leq 1$ and $\operatorname{Re}(\mu_j) \geq 0$. Write

$$-\frac{L'}{L}(s, \pi) = \sum_{n=1}^{\infty} \frac{\lambda_{\pi}(n) \Lambda(n)}{n^s},$$

where $\lambda_{\pi}(n) = 0$ unless $n = p^k$ is a prime power, when it equals $\sum_{j=1}^m \alpha_{j,\pi}(p)^k$. We require the existence of two constants $A_0, A \geq 1$ such that for all $x \geq 1$ the inequality

$$\sum_{x < n < ex} \frac{|\lambda_{\pi}(n)|^2}{n} \Lambda(n) \leq A^2 + \frac{A_0}{\log ex} \quad (76)$$

is satisfied; note that the Ramanujan conjecture would imply this with $A = m$ and $A_0 \ll m^2$. The condition on the parameters μ_j is that $\operatorname{Re}(\mu_j) \geq -1 + \delta_m$ for some $\delta_m > 0$ and all j . If we define the analytic conductor of π to be

$$C(\pi) = N \prod_{j=1}^m (1 + |\mu_j|),$$

Soundararajan proves the following.

Theorem 16. *Under the assumptions on L stated above,*

$$L(1/2, \pi) \ll \frac{C(\pi)^{1/4}}{(\log C(\pi))^{1-\epsilon}},$$

where the implied constant depends on m, ϵ, A_0, A and δ_m .

We may prove theorem 5 by applying this result to the L values $L(1/2, \text{sym}^2 \pi \otimes \pi')$ and $L(1/2 + it, \text{sym}^2 \pi \otimes \lambda_{-m})$ appearing in equations (69) and (74), once we have established that the L functions satisfy the necessary hypotheses. While they are L functions over F , they may be considered as being over \mathbb{Q} by formal base change. We begin with $L(s, \text{sym}^2 \pi \otimes \lambda_{-m})$; if $L(s, \pi)$ has the Euler product expansion

$$L(s, \pi) = \prod_{\mathfrak{p}} \left(1 - \frac{\alpha_{\pi}(\mathfrak{p})}{N\mathfrak{p}^s}\right)^{-1} \left(1 - \frac{\beta_{\pi}(\mathfrak{p})}{N\mathfrak{p}^s}\right)^{-1},$$

$L(s, \text{sym}^2 \pi \otimes \lambda_{-m})$ is given by the Euler product

$$L(s, \text{sym}^2 \pi \otimes \lambda_{-m}) = \prod_{\mathfrak{p}} \left(1 - \frac{\alpha_{\pi}(\mathfrak{p})^2 \lambda_{-m}(\mathfrak{p})}{N\mathfrak{p}^s}\right)^{-1} \left(1 - \frac{\lambda_{-m}(\mathfrak{p})}{N\mathfrak{p}^s}\right)^{-1} \left(1 - \frac{\beta_{\pi}(\mathfrak{p})^2 \lambda_{-m}(\mathfrak{p})}{N\mathfrak{p}^s}\right)^{-1}.$$

Recall our assumption that $|\alpha_{\pi}(\mathfrak{p})| = |\beta_{\pi}(\mathfrak{p})| = 1$. The Archimedean factor of this function is

$$L_{\infty}(s, \text{sym}^2 \pi \otimes \lambda_{-m}) = \prod_j L_{\infty, j}(s, \text{sym}^2 \pi \otimes \lambda_{-m}),$$

where

$$L_{\infty, j}(s, \text{sym}^2 \pi \otimes \lambda_{-m}) = \Gamma_{\mathbb{R}}(s + \beta(m, j) + 1) \Gamma_{\mathbb{R}}(s + \beta(m, j) + k_j - 1) \Gamma_{\mathbb{R}}(s + \beta(m, j) + k_j)$$

for ν_j real and

$$\begin{aligned} L_{\infty, j}(s, \text{sym}^2 \pi \otimes \lambda_{-m}) &= \Gamma_{\mathbb{C}}(s + k_j/2 + \beta(m, j))^2 \Gamma_{\mathbb{C}}(s + \beta(m, j)) \\ &= \Gamma_{\mathbb{R}}(s + k_j/2 + \beta(m, j))^2 \Gamma_{\mathbb{R}}(s + k_j/2 + \beta(m, j) + 1)^2 \\ &\quad \Gamma_{\mathbb{R}}(s + \beta(m, j)) \Gamma_{\mathbb{R}}(s + \beta(m, j) + 1) \end{aligned}$$

for ν_j complex. In particular, it can be seen that all μ_j satisfy $\text{Re}(\mu_j) \geq 0$. By work of Shimura, it is known that the completed L function $\Lambda(s, \text{sym}^2 \pi \otimes \lambda_{-m})$ admits an analytic continuation to the whole complex plane and satisfies the functional equation

$$\Lambda(s, \text{sym}^2 \pi \otimes \lambda_{-m}) = \Lambda(1 - s, \text{sym}^2 \pi \otimes \lambda_m).$$

Therefore the L function we consider satisfies all the hypotheses of Soundararajan's theorem as an Euler product over F , and it will continue to do so when considered as a product over \mathbb{Q} - in particular, it will continue to satisfy the Ramanujan bound. To apply the theorem to the value $L(1/2 + it, \text{sym}^2 \pi \otimes \lambda_{-m})$ we replace $L(s, \text{sym}^2 \pi \otimes \lambda_{-m})$ with the shifted function $L(s + it, \text{sym}^2 \pi \otimes \lambda_{-m})$, which still satisfies all the hypotheses and whose analytic conductor is now $\ll Nk^2(1 + |t| + \|m\|)^{3n}$, to obtain

$$L(1/2 + it, \text{sym}^2 \pi \otimes \lambda_{-m}) \ll \frac{Nk^{1/2}(1 + |t| + \|m\|)^{3n/4}}{(\log Nk)^{1-\epsilon}}.$$

Turning now to $L(1/2, \text{sym}^2 \pi \otimes \pi')$, let $L(s, \pi')$ have the Euler product

$$L(s, \pi') = \prod_{\mathfrak{p}} \left(1 - \frac{\alpha'_{\pi}(\mathfrak{p})}{N\mathfrak{p}^s}\right)^{-1} \left(1 - \frac{\beta'_{\pi}(\mathfrak{p})}{N\mathfrak{p}^s}\right)^{-1}$$

so that $L(s, \text{sym}^2 \pi \otimes \pi')$ has the product expansion

$$\begin{aligned} L(s, \text{sym}^2 \pi \otimes \pi') = \prod_{\mathfrak{p}} & \left(1 - \frac{\alpha_{\pi}(\mathfrak{p})^2 \alpha'_{\pi}(\mathfrak{p})}{N\mathfrak{p}^s}\right)^{-1} \left(1 - \frac{\alpha'_{\pi}(\mathfrak{p})}{N\mathfrak{p}^s}\right)^{-1} \left(1 - \frac{\beta_{\pi}(\mathfrak{p})^2 \alpha'_{\pi}(\mathfrak{p})}{N\mathfrak{p}^s}\right)^{-1} \\ & \left(1 - \frac{\alpha_{\pi}(\mathfrak{p})^2 \beta'_{\pi}(\mathfrak{p})}{N\mathfrak{p}^s}\right)^{-1} \left(1 - \frac{\beta'_{\pi}(\mathfrak{p})}{N\mathfrak{p}^s}\right)^{-1} \left(1 - \frac{\beta_{\pi}(\mathfrak{p})^2 \beta'_{\pi}(\mathfrak{p})}{N\mathfrak{p}^s}\right)^{-1}. \end{aligned}$$

This L function does not necessarily satisfy the Ramanujan bound because we are not assuming it for the representation π' , however because π' is fixed the weaker estimate (76) will still hold by Rankin-Selberg theory applied to π' . The Archimedean factor $L_{\infty,j}(s, \text{sym}^2 \pi \otimes \pi')$ at a real place is

$$L_{\infty,j}(s, \text{sym}^2 \pi \otimes \pi') = \Gamma_{\mathbb{R}}(s + k_j - 1 \pm ir'_j) \Gamma_{\mathbb{R}}(s + k_j \pm ir'_j) \Gamma_{\mathbb{R}}(s \pm ir'_j) \Gamma_{\mathbb{R}}(s + 1 \pm ir'_j),$$

and at a complex place is

$$\begin{aligned} L_{\infty,j}(s, \text{sym}^2 \pi \otimes \pi') &= \Gamma_{\mathbb{C}}(s + k_j/2 \pm ir'_j/2)^2 \Gamma_{\mathbb{C}}(s \pm ir'_j/2) \\ &= \Gamma_{\mathbb{R}}(s + k_j/2 \pm ir'_j/2)^2 \Gamma_{\mathbb{R}}(s + k_j/2 + 1 \pm ir'_j/2)^2 \\ &\quad \Gamma_{\mathbb{R}}(s \pm ir'_j/2) \Gamma_{\mathbb{R}}(s + 1 \pm ir'_j/2). \end{aligned}$$

The required bound $\text{Re}(\mu_j) \geq -1 + \delta$ now follows from the trivial bounds $\text{Im}(r'_j) \leq 1/2$ for ν_j real and $\text{Im}(r'_j) \leq 1$ for ν_j complex. It is known by the work of Garrett [7] that the completed L function is entire in \mathbb{C} , and its value at s is equal to its value at $1 - s$. It only remains to show that $L(s, \text{sym}^2 \pi \otimes \pi')$ satisfies the weak Ramanujan bound (76) as a Dirichlet series over \mathbb{Q} . We have

$$-\frac{L'}{L}(s, \text{sym}^2\pi \otimes \pi') = \sum_{\mathfrak{p}} \log N\mathfrak{p} \sum_{n=1}^{\infty} \frac{(\alpha'_{\pi}(\mathfrak{p}) + \beta'_{\pi}(\mathfrak{p}))(\alpha_{\pi}^{2n}(\mathfrak{p}) + 1 + \beta_{\pi}^{2n}(\mathfrak{p}))}{N\mathfrak{p}^{ns}},$$

and L satisfying the weak Ramanujan bound as a Dirichlet series over \mathbb{Q} is equivalent to the bound

$$\sum_{\substack{\mathfrak{p}, n \\ x < N\mathfrak{p}^n \leq ex}} \log N\mathfrak{p} \frac{|(\alpha'_{\pi}(\mathfrak{p}) + \beta'_{\pi}(\mathfrak{p}))(\alpha_{\pi}^{2n}(\mathfrak{p}) + 1 + \beta_{\pi}^{2n}(\mathfrak{p}))|^2}{N\mathfrak{p}^n} \leq A^2 + \frac{A_0}{\log(ex)}.$$

Applying the Ramanujan bound $|\alpha_{\pi}(\mathfrak{p})| = |\beta_{\pi}(\mathfrak{p})| = 1$, we only need to show that

$$\sum_{\substack{\mathfrak{p}, n \\ x < N\mathfrak{p}^n \leq ex}} \log N\mathfrak{p} \frac{|\alpha'_{\pi}(\mathfrak{p}) + \beta'_{\pi}(\mathfrak{p})|^2}{N\mathfrak{p}^n} \leq A^2 + \frac{A_0}{\log(ex)}$$

for all $x \geq 1$, where A and A_0 are constants which are allowed to depend on π' . This follows from Rankin-Selberg theory for $L(s, \pi' \times \tilde{\pi}')$, whose logarithmic derivative is

$$-\frac{L'}{L}(s, \pi' \times \tilde{\pi}') = \sum_{\mathfrak{p}} \sum_{n=1}^{\infty} \log N\mathfrak{p} \frac{|\alpha'_{\pi}(\mathfrak{p}) + \beta'_{\pi}(\mathfrak{p})|^2}{N\mathfrak{p}^{ns}}.$$

Because $L(s, \pi' \times \tilde{\pi}')$ has a classical zero-free region $\text{Re}(s) \geq 1 - c'_{\pi}/\log(1 + |t|)$, it follows in the same way as the proof of the prime number theorem that

$$\sum_{\substack{\mathfrak{p}, n \\ x < N\mathfrak{p}^n \leq ex}} \frac{\log N\mathfrak{p} |\alpha'_{\pi}(\mathfrak{p}) + \beta'_{\pi}(\mathfrak{p})|^2}{N\mathfrak{p}^{ns}} = 1 + O_{\pi'}\left(\frac{1}{\log(ex)}\right),$$

from which (76) follows. As $L(s, \text{sym}^2\pi \otimes \pi')$ has analytic conductor $\ll Nk^4$, we may now apply the weak subconvex estimate to $L(1/2, \text{sym}^2\pi \otimes \pi')$ to complete the proof of theorem 5.

9 Conclusion of Proof

We now conclude the proof of theorem 3, by presenting the way in which theorems 5 and 4 may be combined as in Holowinsky and Soundararajan's paper [15]. This relies on a lower bound for $L(1, \text{sym}^2\pi)$ and a relation between this value and the quantity $M_k(\pi)$ appearing in theorem 4. We first consider the symmetric square L function $L(s, \text{sym}^2\pi)$, whose definition and basic analytic properties were given in section 8.2, and collect some important results on it due to work of Gelbart and Jacquet [8], Hoffstein and Lockhart [12], and Goldfeld,

Hoffstein and Lockhart [9]. The lower bound we shall use for $L(1, \text{sym}^2 \pi)$ is proved using the symmetric square lift of Gelbart and Jacquet [8] from $GL(2)$ to $GL(3)$, which shows that $L(s, \text{sym}^2 \pi)$ is the standard L function of a cuspidal automorphic form on $GL(3)$. Using the Rankin-Selberg convolution for this form, one may then establish a standard zero-free region for $L(s, \text{sym}^2 \pi)$. For instance, using Theorem 5.42 (or Theorem 5.44) of Iwaniec and Kowalski [17] one may show that for some constant $c > 0$ the region

$$\mathcal{R} = \left\{ s = \sigma + it : \sigma \geq 1 - \frac{c}{\log \|k\|(1 + |t|)} \right\}$$

contains no zero of $L(s, \text{sym}^2 \pi)$ other than possibly a simple real zero. This exceptional zero is ruled out by work of Hoffstein and Lockhart [12] (see the appendix by Goldfeld, Hoffstein and Lockhart [9]), who show that there is an effectively computable choice of $c > 0$ such that \mathcal{R} is totally zero free. Furthermore, Goldfeld, Hoffstein and Lockhart [9] show that

$$L(1, \text{sym}^2 \pi) \gg \frac{1}{\log \|k\|}. \quad (77)$$

The first consequence of this lower bound is the following.

Lemma 17. *For any $t \in \mathbb{R}$ and $m \in \mathbb{Z}^{n-1}$, we have*

$$|L(s, \text{sym}^2 \pi \otimes \lambda_m)| \ll \frac{Nk^{1/2}(1 + |t| + \|m\|)^{3n/4}}{(\log \|k\|)^{1-\epsilon}}.$$

Therefore the quantity $R_k(\pi)$ appearing in theorem 4 satisfies

$$R_k(\pi) \ll \frac{(\log \|k\|)^\epsilon}{(\log \|k\|)L(1, \text{sym}^2 \pi)} \ll (\log \|k\|)^\epsilon.$$

Proof. The first inequality follows from weak subconvexity, and is proven in section 8.2. The second bound follows immediately by substituting the first in the formula for $R_k(\pi)$ and applying the lower bound (77) for $L(1, \text{sym}^2 \pi)$. \square

The relationship between $M_k(\pi)$ and $L(1, \text{sym}^2 \pi)$ we shall use is based on the following lemma.

Lemma 18. *We have*

$$L(1, \text{sym}^2 \pi) \gg (\log \log \|k\|)^{-3} \exp \left(\sum_{N\mathfrak{p} \leq \|k\|} \frac{\lambda_\pi(\mathfrak{p}^2)}{N\mathfrak{p}} \right).$$

The proof of this over \mathbb{Q} in [15] may be extended to a number field; the only modification is generalising the asymptotic $\sum_{p \leq x} 1/p = \log \log x + O(1)$ to $\sum_{N\mathfrak{p} \leq x} 1/N\mathfrak{p} = \log \log x + O(1)$. Lemma 18 gives us the required estimate for $M_k(\pi)$ below.

Lemma 19. *We have*

$$M_k(\pi) \ll (\log \|k\|)^{1/6} (\log \log \|k\|)^{9/2} L(1, \text{sym}^2 \pi)^{1/2}.$$

Proof. From the inequality $2|x| \leq \frac{2}{3} + \frac{3}{2}x^2$ and the Hecke relations, we obtain

$$2 \sum_{N\mathfrak{p} \leq \|k\|} \frac{|\lambda_\pi(\mathfrak{p})|}{N\mathfrak{p}} \leq \frac{2}{3} \sum_{N\mathfrak{p} \leq \|k\|} \frac{1}{N\mathfrak{p}} + \frac{3}{2} \sum_{N\mathfrak{p} \leq \|k\|} \frac{\lambda_\pi(\mathfrak{p})^2}{N\mathfrak{p}} = \frac{13}{6} \sum_{N\mathfrak{p} \leq \|k\|} \frac{1}{N\mathfrak{p}} + \frac{3}{2} \sum_{N\mathfrak{p} \leq \|k\|} \frac{\lambda_\pi(\mathfrak{p}^2)}{N\mathfrak{p}}.$$

Using lemma 18 and $\sum_{N\mathfrak{p} \leq x} 1/N\mathfrak{p} = \log \log x + O(1)$, the lemma follows. \square

We may now prove the decay of $\langle \phi F_k, F_k \rangle$ for ϕ a Hecke-Maass cusp form or pure incomplete Eisenstein series. We consider the Maass case first. If $L(1, \text{sym}^2 \pi) \geq (\log \|k\|)^{-7/15}$, then theorem 5 gives $\langle \phi F_k, F_k \rangle \ll (\log \|k\|)^{-1/30+\epsilon}$. Otherwise, from lemma 19 we have that $M_k(\pi) \ll (\log \|k\|)^{-1/15+\epsilon}$, and now theorem 4 gives $\langle \phi F_k, F_k \rangle \ll (\log \|k\|)^{-1/30+\epsilon}$. Therefore the bound of theorem 3 holds in either case.

In the Eisenstein case, we begin by showing how theorem 5 may be used to treat pure incomplete Eisenstein series in the cases where $L(1, \text{sym}^2 \pi)$ is large. By Mellin inversion, we may write

$$E(\psi, m|z) = \frac{1}{2\pi i} \int_{(\sigma)} \Psi(-s) E(s, m, z) ds,$$

where Ψ is the Mellin transform of ψ . We may move the line of integration to $\sigma = 1/2$ to obtain

$$E(\psi, m|z) = \frac{1}{\text{Vol}(Y)} \langle E(\psi, m|z), 1 \rangle + \frac{1}{2\pi i} \int_{(1/2)} \Psi(-s) E(s, m, z) ds,$$

and so

$$\langle E(\psi, m|z) F_k, F_k \rangle = \frac{1}{\text{Vol}(Y)} \langle E(\psi, m|z), 1 \rangle + \frac{1}{2\pi i} \int_{(1/2)} \Psi(-s) \langle E(s, m, z) F_k, F_k \rangle ds. \quad (78)$$

We now apply theorem 5 to obtain the bound

$$\begin{aligned} \int_{(1/2)} \Psi(-s) \langle E(s, m, z) F_k, F_k \rangle ds &\ll \int_{\mathbb{R}} |\Psi(-1/2 - it)| \frac{(1 + |t| + \|m\|)^{2n}}{(\log \|k\|)^{1-\epsilon} L(1, \text{sym}^2 \pi)} ds \\ &\ll \frac{(\log \|k\|)^{-1+\epsilon}}{L(1, \text{sym}^2 \pi)}. \end{aligned}$$

It follows by substituting this in (78) that

$$\left| \langle E(\psi, m|z)F_k, F_k \rangle - \frac{1}{\text{Vol}(Y)} \langle E(\psi, m|z), 1 \rangle \right| \ll \frac{(\log \|k\|)^{-1+\epsilon}}{L(1, \text{sym}^2 \pi)}.$$

Therefore if $L(1, \text{sym}^2 \pi) \geq (\log \|k\|)^{-13/15}$, we obtain the bound of theorem 3. If $L(1, \text{sym}^2 \pi) < (\log \|k\|)^{-13/15}$, lemma 19 gives $M_k(\pi) \ll (\log \|k\|)^{-4/15+\epsilon}$. Applying proposition 4 with the bound on $R_k(\pi)$ provided by lemma 17, we have

$$\left| \langle E(\psi, m|z)F_k, F_k \rangle - \frac{1}{\text{Vol}(Y)} \langle E(\psi, m|z), 1 \rangle \right| \ll (\log \|k\|)^\epsilon M_k(\pi)^{1/2} \ll (\log \|k\|)^{-2/15+\epsilon},$$

and so the bound of theorem 3 hold in this case also.

10 Equidistribution of Zero Currents

This section contains the proof of theorem 6 on the equidistribution of the zero divisors of holomorphic Hecke modular forms. The proof is based on ideas from complex potential theory, which may be described in the general context of of a compact complex manifold M with a positive holomorphic line bundle L . Suppose that L has been equipped with a Hermitian metric h , and let $\omega = c_1(h)$ be the associated Kähler form. If $s_N \in H^0(M, L^N)$ are a sequence of L^2 normalised sections of L^N whose mass becomes equidistributed on M , potential theory may be used to show that their normalised zero divisors $\frac{1}{N}Z_N$ tend weakly to ω in the sense of currents described in section 3.1. This was first discovered by Nonnenmacher and Vorros [28] in the context of quantum maps on tori, and extended in the generality described here by Schiffman and Zelditch [38]. If we now let \mathbb{H}^n denote the product of n upper half planes and let L_k be the line bundle of differentials of the form $f(z) \otimes_i dz_i^{k_i/2}$ on \mathbb{H}^n , or its quotient by Γ , theorem 6 is thus an extension of the result in [38] to the bundle L_k over the noncompact manifold Y . To prove it we shall apply the argument of Schiffman and Zelditch, adding the adjustments of Rudnick [29] to deal with the cusp.

We may give L_k the natural Hermitian inner product $\| \otimes dz_i^{k_i/2} \|^2 = y^k$, whose associated Kähler form ω is

$$\begin{aligned} \omega &= \frac{-i}{2\pi} \partial \bar{\partial} \log y^k \\ &= \frac{1}{4\pi} \sum k_i y_i^{-2} dx_i \wedge dy_i. \end{aligned}$$

If f is a holomorphic modular form of weight k , $f \otimes dz_i^{k_i/2}$ is then a section of L_k with $\|f \otimes dz_i^{k_i/2}\|^2 = |f(z)|^2 y^k$. We let Z_f be the zero divisor of f on Y , and \tilde{Z}_f its pullback to \mathbb{H}^n . For $\phi \in A^{n-1, n-1}(\mathbb{H}^n)$ smooth and compactly supported, let

$$F_\phi = \sum_{\gamma \in \Gamma} \gamma^* \phi$$

be its symmetrisation under Γ . If f_N are a sequence of modular forms of weight Nk as in theorem 6, we shall compare $\frac{1}{N}Z_N$ and ω by testing them against the differential forms F_ϕ using the following lemma.

Lemma 20. *If f is a holomorphic modular form of weight k on Y ,*

$$\int_{Z_f} F_\phi = \int_Y F_\phi \wedge \omega + \frac{i}{\pi} \int_{\mathbb{H}^n} \log(y^{k/2}|f(z)|) \partial \bar{\partial} \phi.$$

Proof. By unfolding F_ϕ , we get

$$\int_{Z_f} F_\phi = \int_{\tilde{Z}_f} \phi. \quad (79)$$

As \tilde{Z}_f is the zero divisor of the global holomorphic function f on \mathbb{H}^n , we may apply the Poincare-Lelong formula to the RHS of (79), obtaining

$$\begin{aligned} \int_{Z_f} F_\phi &= \frac{i}{\pi} \int_{\mathbb{H}^n} \log |f(z)| \partial \bar{\partial} \phi \\ &= -\frac{i}{\pi} \int_{\mathbb{H}^n} \log y^{k/2} \partial \bar{\partial} \phi + \frac{i}{\pi} \int_{\mathbb{H}^n} \log(y^{k/2}|f(z)|) \partial \bar{\partial} \phi. \end{aligned}$$

After integration by parts the first term becomes

$$-\frac{i}{\pi} \int_{\mathbb{H}^n} \partial \bar{\partial} \log y^{k/2} \phi = \int_{\mathbb{H}^n} \omega \wedge \phi$$

and may be refolded to $\int_Y \omega \wedge F_\phi$, which completes the proof. \square

After applying lemma 20 to $\frac{1}{N}Z_N$, we are left with proving that $\frac{1}{N} \log(y^{Nk/2}|f_N(z)|) \xrightarrow{w^*} 0$ locally everywhere. As in [29, 38] this will follow from the plurisubharmonicity of $\log |f_N|$ and the equidistribution result $y^{Nk}|f_N(z)|^2 \xrightarrow{w^*} c$, once we know that $\frac{1}{N} \log(y^{Nk/2}|f_N(z)|)$ is bounded above and has lim sup equal to 0, and that both properties hold locally uniformly. Both of these are provided by the following lemma and the assumption (which we may clearly make) that the f_N are L^2 normalised.

Lemma 21. *Let f be a Hecke cusp form of weight k for Γ . Then uniformly for z in compact subsets of \mathbb{H}^n ,*

$$\frac{y^k |f(z)|^2}{\|f\|^2} \ll_\Gamma N k^{5/2+\epsilon}. \quad (80)$$

Proof. Assume $\|f\|^2 = 1$. We shall bound $|f|$ using its Fourier expansion

$$f(z) = \sum_{\xi > 0} a_f(\xi) e(\text{tr}(\xi \kappa z)),$$

and the proportionality relation $a_f(\xi) = \lambda_\pi(\xi) a_f(1) \xi^{(k-1)/2}$ with $\lambda_\pi(\xi) \ll N \xi^\epsilon$. Applying these and the normalisation of $a_f(1)$ from (21), we have

$$\begin{aligned} y^{k/2} |f(z)| &\leq \sum_{\xi > 0} |a_f(\xi)| y^{k/2} \exp(-2\pi \text{tr}(\xi \kappa y)) \\ &\ll \kappa^{k/2} N k^\epsilon \prod_{i=1}^n \frac{(4\pi)^{k_i/2}}{\Gamma(k_i)^{1/2}} \sum_{\xi > 0} \xi^{(k-1)/2+\epsilon} y^{k/2} \exp(-2\pi \text{tr}(\xi \kappa y)) \\ &= N k^\epsilon \prod_{i=1}^n \frac{1}{\Gamma(k_i)^{1/2}} \sum_{\xi > 0} N \xi^{-1/2+\epsilon} (4\pi \xi \kappa y)^{k/2} \exp(-2\pi \text{tr}(\xi \kappa y)) \\ &\leq N k^\epsilon \prod_{i=1}^n \frac{1}{\Gamma(k_i)^{1/2}} \sum_{\xi > 0} (4\pi \xi \kappa y)^{k/2} \exp(-2\pi \text{tr}(\xi \kappa y)). \end{aligned}$$

We define $g_i(x) = x^{k_i/2} e^{-x/2}$ and let $g : \mathbb{R}_+^n \rightarrow \mathbb{R}$ be the product function. If we define L to be the semi-lattice $4\pi \kappa y \mathcal{O} \cap \mathbb{R}_+^n$, the upper bound above may be written

$$y^{k/2} |f(z)| \ll N k^\epsilon \prod_{i=1}^n \frac{1}{\Gamma(k_i)^{1/2}} \sum_{x \in L} g(x). \quad (81)$$

We now apply a lemma bounding the sum in (81) in terms of various integrals of g . Suppose g_i is increasing for $x < t_i$ and decreasing for $x > t_i$. Let \mathcal{P} be the set of subsets of $\{1, \dots, n\}$, and for $S \in \mathcal{P}$ define the subspace $H_S \in \mathbb{R}_+^n$ by

$$H_S = \{x \in \mathbb{R}_+^n \mid x_i = t_i, i \in S\}.$$

We then have the following bound (whose proof we omit) on $\sum_{x \in L} g(x)$.

Lemma 22.

$$\begin{aligned} \sum_{x \in L} g(x) &\ll \sum_{S \in \mathcal{P}} \int_{H_S} g dv \\ &= \prod_{i=1}^n \left(\int_{\mathbb{R}^+} g(t) dt + g(t_i) \right), \end{aligned}$$

where the implied constant is bounded in compact families of lattices L .

Applying this to (81) gives

$$\begin{aligned}
y^{k/2}|f(z)| &\ll Nk^\epsilon \prod_{i=1}^n \frac{1}{\Gamma(k_i)^{1/2}} \left(\int_{\mathbb{R}^+} x^{k_i/2} e^{-x/2} dx + k_i^{k_i/2} e^{-k_i/2} \right) \\
&= Nk^\epsilon \prod_{i=1}^n \frac{1}{\Gamma(k_i)^{1/2}} \left(2^{k_i/2+1} \Gamma(k_i/2 + 1) + k_i^{k_i/2} e^{-k_i/2} \right) \\
&\ll Nk^\epsilon \prod_{i=1}^n (k_i^{5/4} + k_i^{-3/4}) \\
&\ll Nk^{5/4+\epsilon}.
\end{aligned}$$

The local uniformity of lemma 22 in L gives the local uniformity of this bound, which completes the proof of lemma 21 and theorem 6. □

11 Appendix

We include here a number of routine calculations that were omitted during the proof of propositions 9 and 13. These are the Fourier expansions of Eisenstein series over mixed number fields, the L^2 normalisations of cohomological automorphic forms, the calculation of the volume of $\mathbb{F}_1^\times / \mathcal{O}_+^\times$ and the verification of the main term picked up in the contour shifts in lemma 12 and equation (47).

11.1 Fourier Expansions of Eisenstein Series

We recall the definition of $E(s, m, z)$ for $s \in \mathbb{C}$ and $m \in \mathbb{Z}^{r-1}$,

$$E(s, m, z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} N(y(\gamma z))^s \lambda_m(y(\gamma z)).$$

We let $s_i = s + \beta(m, i)/\delta_i$, so that this may be rewritten

$$E(s, m, z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \prod_{i=1}^r y_i(\gamma z)^{\delta_i s_i}.$$

The map sending $\gamma \in \Gamma_\infty \backslash \Gamma$ to its lower two entries is a bijection from $\Gamma_\infty \backslash \Gamma$ to the set of pairs $\{c, d\}$ of relatively prime elements of \mathcal{O} modulo \mathcal{O}^\times , and $y_i(\gamma z)$ may be expressed in terms of this pair as

$$\begin{aligned}
y_i(\gamma z) &= \frac{y_i}{|c_i z_i + d_i|^2}, & i \leq r_1 \\
y_i(\gamma z) &= \frac{y_i}{|c_i|^2 y_i^2 + |c_i x_i + d_i|^2}, & i > r_1.
\end{aligned}$$

Therefore if we define $F(s, m, z)$ by

$$F(s, m, z) = \sum_{\{c, d\}} \prod_{i \leq r_1} \frac{y_i^{s_i}}{|c_i z_i + d_i|^{2s_i}} \prod_{i > r_1} \frac{y_i^{2s_i}}{(|c_i|^2 y_i^2 + |c_i x_i + d_i|^2)^{2s_i}}, \quad (82)$$

where the sum is over all pairs $\{c, d\}$ modulo \mathcal{O}^\times , we have $F(s, m, z) = \zeta(2s, \lambda_{-2m})E(s, m, z)$. The ξ th Fourier coefficient of $F(s, m, z)$ is the integral

$$a_\xi(s, m, y) = \frac{2^{r_2}}{\sqrt{|D|}} \int_{\mathbb{F}/\mathcal{O}} F(x + jy, s, m) e(-\text{tr}(\xi \kappa x)) dx. \quad (83)$$

We begin by collecting the terms in (82) with $c = 0$ to write

$$F(s, m, z) = Ny^s \lambda_m(y) \zeta(2s, \lambda_{-2m}) + \sum_{(c)} \frac{Ny^s \lambda_m(y)}{Nc^{2s} \lambda_{2m}(c)} \sum_{d \bmod (c)} \sum_{\alpha \in \mathcal{O}} \prod_{i \leq r_1} \frac{1}{|z_i + \frac{d_i}{c_i} + \alpha_i|^{2s_i}} \prod_{i > r_1} \frac{1}{(y_i^2 + |x_i + \frac{d_i}{c_i} + \alpha_i|^2)^{2s_i}}.$$

Substituting this into (83) and unfolding over \mathcal{O} , we express $a_\xi(s, m, y)$ as

$$a_\xi(s, m, y) = \delta_{\xi 0} Ny^s \lambda_m(y) \zeta(2s, \lambda_{-2m}) + \frac{2^{r_2}}{\sqrt{|D|}} \sum_{(c)} \frac{Ny^s \lambda_m(y)}{Nc^{2s} \lambda_{2m}(c)} \sum_{d \bmod (c)} \prod_{i \leq r_1} \int_{\mathbb{R}} \frac{e(-\xi_i \kappa_i x_i) dx_i}{|z_i + \frac{d_i}{c_i}|^{2s_i}} \prod_{i > r_1} \int_{\mathbb{C}} \frac{e(-\text{tr}(\xi_i \kappa_i x_i)) dx_i}{(y_i^2 + |x_i + \frac{d_i}{c_i}|^2)^{2s_i}}.$$

We first consider the case $\xi = 0$.

$$\begin{aligned} a_0(s, m, y) &= Ny^s \lambda_m(y) \zeta(2s, \lambda_{-2m}) + \frac{2^{r_2}}{\sqrt{|D|}} \sum_{(c)} \frac{Ny^s \lambda_m(y)}{Nc^{2s} \lambda_{2m}(c)} \sum_{d \bmod (c)} \\ &\quad \prod_{i \leq r_1} \int_{\mathbb{R}} \frac{dx_i}{|z_i + \frac{d_i}{c_i}|^{2s_i}} \prod_{i > r_1} \int_{\mathbb{C}} \frac{dx_i}{(y_i^2 + |x_i + \frac{d_i}{c_i}|^2)^{2s_i}} \\ &= Ny^s \lambda_m(y) \zeta(2s, \lambda_{-2m}) + \frac{2^{r_2}}{\sqrt{|D|}} \sum_{(c)} \frac{Ny^{1-s} \lambda_{-m}(y)}{Nc^{2s-1} \lambda_{2m}(c)} \\ &\quad \prod_{i \leq r_1} \int_{\mathbb{R}} \frac{dx_i}{(1 + x_i^2)^{s_i}} \prod_{i > r_1} \int_{\mathbb{C}} \frac{dx_i}{(1 + |x_i|^2)^{2s_i}}. \\ &= Ny^s \lambda_m(y) \zeta(2s, \lambda_{-2m}) + \frac{\pi^{n/2}}{\sqrt{|D|}} Ny^{1-s} \lambda_{-m}(y) \zeta(2s-1, \lambda_{-2m}) \\ &\quad \prod_{i \leq r_1} \frac{\Gamma(s + \beta(m, i) - 1/2)}{\Gamma(s + \beta(m, i))} \prod_{i > r_1} \frac{2}{2s + \beta(m, i) - 1}. \end{aligned}$$

On dividing through by $\zeta(2s, \lambda_{-2m})$, this agrees with the expression given in section 6.1. When $\xi \neq 0$, we have

$$\begin{aligned}
a_\xi(s, m, y) &= \frac{2^{r_2}}{\sqrt{|D|}} \sum_{(c)} \frac{Ny^s \lambda_m(y)}{Nc^{2s} \lambda_{2m}(c)} \sum_{d \bmod (c)} \\
&\quad \prod_{i \leq r_1} \int_{\mathbb{R}} \frac{e(-\xi_i \kappa_i x_i) dx_i}{|z_i + \frac{d_i}{c_i}|^{2s_i}} \prod_{i > r_1} \int_{\mathbb{C}} \frac{e(-\text{tr}(\xi_i \kappa_i x_i)) dx_i}{(y_i^2 + |x_i + \frac{d_i}{c_i}|^2)^{2s_i}} \\
&= \frac{2^{r_2}}{\sqrt{|D|}} \sum_{(c)} \frac{Ny^{1-s} \lambda_{-m}(y)}{Nc^{2s} \lambda_{2m}(c)} \sum_{d \bmod (c)} e(\text{tr}(\frac{\xi \kappa d}{c})) \\
&\quad \prod_{i \leq r_1} \int_{\mathbb{R}} \frac{e(-\xi_i \kappa_i y_i x_i) dx_i}{(1 + x_i^2)^{s_i}} \prod_{i > r_1} \int_{\mathbb{C}} \frac{e(-\text{tr}(\xi_i \kappa_i y_i x_i)) dx_i}{(1 + |x_i|^2)^{2s_i}}.
\end{aligned}$$

The integral at real places is equal to

$$\frac{2\pi^{s_i}}{\Gamma(s_i)} (\xi_i \kappa_i y_i)^{s_i-1/2} K_{s_i-1/2}(2\pi |\xi_i \kappa_i| y_i),$$

and the integral at complex places may be calculated in the following way as in [31].

$$\begin{aligned}
\int_{\mathbb{C}} \frac{e(-\text{tr}(\xi_i \kappa_i y_i x_i)) dx_i}{(1 + |x_i|^2)^{2s_i}} &= \int_0^\infty \int_0^{2\pi} \frac{e(-2y_i r |\xi_i \kappa_i| \sin(\theta + \alpha))}{(r^2 + 1)^{2s_i}} r d\theta dr \\
&= \int_0^\infty \frac{r}{(r^2 + 1)^{2s_i}} \int_0^{2\pi} e(-2y_i r |\xi_i \kappa_i| \sin \theta) d\theta dr \\
&= \int_0^\infty \frac{J_0(4\pi r |\xi_i \kappa_i| y_i)}{(r^2 + 1)^{2s_i}} dr \\
&= \frac{(4\pi |\xi_i \kappa_i| y_i)^{2s_i-1}}{\Gamma(2s_i) 2^{2s_i-1}} K_{2s_i-1}(4\pi |\xi_i \kappa_i| y_i).
\end{aligned}$$

(See [10] for the evaluation of the final integral.) It can be seen from this that the final form of $a_\xi(s, m, y)$ is the product of a collection of Bessel functions and Gamma factors which agree with the formula for $E(s, m, z)$ of section 6.1, together with a constant term and a power of y which are given below

$$\frac{2^{r_2}}{\sqrt{|D|}} Ny^{1-s} \lambda_{-m}(y) \sum_{(c)} \sum_{d \bmod (c)} e(\text{tr}(\frac{\xi \kappa d}{c})) \prod_{i \leq r_1} 2\pi^{s_i} (\xi_i \kappa_i y_i)^{s_i-1/2} \prod_{i > r_1} \frac{(4\pi |\xi_i \kappa_i| y_i)^{2s_i-1}}{2^{2s_i-1}}.$$

The power of y simplifies to \sqrt{Ny} , while the constant may be simplified as

$$\begin{aligned}
& \frac{2^{r_2}}{\sqrt{|D|}} \sum_{(c)} \sum_{d \bmod (c)} e(\text{tr}(\frac{\xi \kappa d}{c})) \prod_{i \leq r_1} 2\pi^{s_i} (\xi_i \kappa_i)^{s_i-1/2} \prod_{i > r_1} (2\pi |\xi_i \kappa_i|)^{2s_i-1} \\
&= \frac{2^r \pi^{ns-r_2}}{\sqrt{|D|}} \sigma_{1-2s, -2m}(\xi \kappa) N(\xi \kappa)^{s-1/2} \lambda_m(\xi \kappa) \prod_{i > r_1} 2^{2s_i-1} \\
&= \frac{2^r \pi^{ns-r_2}}{\sqrt{|D|}} \sigma_{1-2s, -2m}(\xi \kappa) N(\delta \xi \kappa)^{s-1/2} \lambda_m(\delta \xi \kappa).
\end{aligned}$$

After dividing through by $\zeta(2s, \lambda_{-2m})$, both of these terms agree with the expression in section 6.1.

11.2 L^2 Normalisations

This section contains the calculation of the L^2 normalisations of the Fourier coefficients of our forms F_k . The normalisations are based on the equation

$$\text{Res}_{s=1} \langle E(s, z) F_k, F_k \rangle = \text{Res}_{s=1} \phi(s) \langle F_k, F_k \rangle, \quad (84)$$

where $E(s, z) = E(s, 0, z)$ and $\phi(s)$ is the scattering coefficient in the constant term. $\text{Res}_{s=1} \phi(s)$ is given by

$$\begin{aligned}
\text{Res}_{s=1} \phi(s) &= \frac{\pi^{n/2}}{\sqrt{|D|}} \frac{\text{Res}_{s=1} \zeta_F(s)}{2\zeta_F(2)} \prod_{i \leq r_1} \frac{\Gamma(1/2)}{\Gamma(1)} \prod_{i > r_1} 2 \\
&= \frac{2^{r_2-1} \pi^{(n+r_1)/2} \text{Res}_{s=1} \zeta_F(s)}{\sqrt{|D|} \zeta_F(2)}.
\end{aligned}$$

We then calculate $\langle E(s, z) F_k, F_k \rangle$ by unfolding and compare the two sides of (84).

$$\begin{aligned}
\langle E(s, z) F_k, F_k \rangle &= \int_{\Gamma_\infty \backslash \mathbb{H}_F} N y^s |F_k|^2 dv \\
&= |a_f(1)|^2 \frac{\sqrt{|D|}}{2^{r_2} \omega_+} \int_{\mathbb{R}_+^r / \mathcal{O}_+^\times} N y^{s-1} \sum_{\eta \in \mathcal{O}^+} N \eta^{-1} |\lambda_\pi(\eta)|^2 |\mathbf{K}_k(\eta \kappa y)|^2 dy^\times \\
&= |a_f(1)|^2 \frac{\sqrt{|D|}}{2^{r_2}} \sum_{(\eta)} N \eta^{-1} |\lambda_\pi(\eta)|^2 \int_{\mathbb{R}_+^r} N y^{s-1} |\mathbf{K}_k(\eta \kappa y)|^2 dy^\times.
\end{aligned}$$

Note that the factor of ω_+ vanished because $\mathcal{O}^+ / \mathcal{O}_+^\times$ counts each ideal with multiplicity ω_+ .

$$\begin{aligned}
\langle E(s, z)F_k, F_k \rangle &= |a_f(1)|^2 \frac{\sqrt{|D|} N \kappa^{1-s}}{2^{r_2}} \sum_{(\eta)} \frac{|\lambda_\pi(\eta)|^2}{N \eta^s} \int_{\mathbb{R}_+^{r_1}} N y^{s-1} |\mathbf{K}_k(y)|^2 dy^\times \\
&= |a_f(1)|^2 \frac{|D|^{s-1/2}}{2^{r_2}} L(s, \text{sym}^2 \pi) \frac{\zeta_F(s)}{\zeta_F(2s)} \int_{\mathbb{R}_+^{r_1}} N y^{s-1} |\mathbf{K}_k(y)|^2 dy^\times.
\end{aligned}$$

We only need the value of the integral at $s = 1$, and to calculate it we expand it as a product over the infinite places. The factor at a real place is

$$\int_0^\infty y^{k_i} \exp(-4\pi k_i y) dy^\times = (4\pi)^{-k_i} \Gamma(k_i).$$

At a complex place, it is

$$\begin{aligned}
&\int_0^\infty y^{k_i+2} \sum_{j=0}^{k_i} \binom{k_i}{j} K_{k_i/2-j}^2(4\pi y) dy^\times \\
&= (4\pi)^{-k_i-2} \sum_{j=0}^{k_i} \binom{k_i}{j} \int_0^\infty y^{k_i+2} K_{k_i/2-j}^2(y) dy^\times \\
&= (4\pi)^{-k_i-2} \frac{2^{k_i-1} \Gamma(1 + k_i/2)^2}{\Gamma(2 + k_i)} \sum_{j=0}^{k_i} \binom{k_i}{j} \Gamma(1 + j) \Gamma(1 + k_i - j) \\
&= (4\pi)^{-k_i-2} \frac{2^{k_i-1} \Gamma(1 + k_i/2)^2}{\Gamma(2 + k_i)} (k_i + 1)! \\
&= 2^{-5} \pi^{-2} (2\pi)^{-k_i} \Gamma(1 + k_i/2)^2.
\end{aligned}$$

Combining these, we have the following expression for $\text{Res}_{s=1} \langle E(s, z)F_k, F_k \rangle$:

$$\begin{aligned}
\text{Res}_{s=1} \langle E(s, z)F_k, F_k \rangle &= |a_f(1)|^2 \frac{\sqrt{|D|}}{2^{6r_2} \pi^{2r_2}} L(1, \text{sym}^2 \pi) \frac{\text{Res}_{s=1} \zeta_F(s)}{\zeta_F(2)} \\
&\quad \prod_{i \leq r_1} (4\pi)^{-k_i} \Gamma(k_i) \prod_{i > r_1} (2\pi)^{-k_i} \Gamma(1 + k_i/2)^2.
\end{aligned}$$

Dividing by $\text{Res}_{s=1} \phi(s)$ we obtain the required relation between $|a_f(1)|^2$ and $\langle F_k, F_k \rangle$,

$$\langle F_k, F_k \rangle = |a_f(1)|^2 \frac{|D| L(1, \text{sym}^2 \pi)}{2^{7r_2-1} \pi^{r_1+3r_2}} \prod_{i \leq r_1} (4\pi)^{-k_i} \Gamma(k_i) \prod_{i > r_1} (2\pi)^{-k_i} \Gamma(1 + k_i/2)^2.$$

11.3 Volume Computations

In this section we compute the volume element in the cusp of Y , and use this with our computation of the residue of $E(s, z)$ to calculate the volume of Y . As in Efrat [4], we shall introduce simplified co-ordinates in the cusp, defined using the matrix A from section 2.2. We define the co-ordinates Y_0, \dots, Y_{r-1} by

$$\begin{pmatrix} \ln Y_0 \\ Y_1 \\ \vdots \\ Y_{r-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 & \dots & 2 \\ e_1^1 & e_2^1 & \dots & e_n^1 \\ \vdots & \vdots & \ddots & \vdots \\ e_1^{n-1} & e_2^{n-1} & \dots & e_n^{n-1} \end{pmatrix} \begin{pmatrix} \ln y_1 \\ \ln y_2 \\ \vdots \\ \ln y_r \end{pmatrix},$$

so that

$$\begin{pmatrix} \ln y_1 \\ \ln y_2 \\ \vdots \\ \ln y_r \end{pmatrix} = \begin{pmatrix} 1/n & \log |\epsilon_1^1| & \dots & \log |\epsilon_{r-1}^1| \\ \vdots & \vdots & \ddots & \vdots \\ 1/n & \log |\epsilon_1^r| & \dots & \log |\epsilon_{r-1}^r| \end{pmatrix} \begin{pmatrix} \ln Y_0 \\ Y_1 \\ \vdots \\ Y_{r-1} \end{pmatrix}.$$

We shall compute the volume form of \mathbb{H}_F with respect to the new system of co-ordinates $Y_0, \dots, Y_{r-1}, x_1, \dots, x_r$. As we are not changing the x -coordinates at all we may omit them from our calculations, and only compute the form $\bigwedge_i dy_i/y_i$ with respect to $\{Y_i\}$. The Jacobian of the change of co-ordinates is

$$\begin{pmatrix} \frac{\partial y_1}{\partial Y_0} \\ \frac{\partial y_2}{\partial Y_0} \\ \vdots \\ \frac{\partial y_r}{\partial Y_0} \end{pmatrix} = \begin{pmatrix} \frac{y_1}{nY_0} & y_1 \log |\epsilon_1^1| & \dots & y_1 \log |\epsilon_{r-1}^1| \\ \vdots & \vdots & \ddots & \vdots \\ \frac{y_r}{nY_0} & y_r \log |\epsilon_1^r| & \dots & y_r \log |\epsilon_{r-1}^r| \end{pmatrix} \begin{pmatrix} \frac{\partial Y_0}{\partial Y_1} \\ \frac{\partial Y_0}{\partial Y_2} \\ \vdots \\ \frac{\partial Y_0}{\partial Y_{r-1}} \end{pmatrix},$$

and we need to calculate its determinant which is

$$\frac{1}{nY_0} \prod_{i=1}^r y_i \det \begin{pmatrix} 1 & \log |\epsilon_1^1| & \dots & \log |\epsilon_{r-1}^1| \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \log |\epsilon_1^r| & \dots & \log |\epsilon_{r-1}^r| \end{pmatrix}.$$

We shall calculate this determinant by a minor expansion along the first column. The absolute value of the determinant of the $(1, i)$ th minor is the regulator R^+ of \mathcal{O}_+^\times times $1/2$ for every complex place we are expanding over. The index of \mathcal{O}_+^\times in \mathcal{O}^\times is $2^{r_1-1+\delta_{r_1^0}}$ so $R^+ = 2^{r_1-1+\delta_{r_1^0}} R$, and the alternating sum of the minors is $(r_1 + 2r_2)2^{r_1-r_2-1+\delta_{r_1^0}} R = n2^{r_1-r_2-1+\delta_{r_1^0}} R$. The expression for dv in terms of our new co-ordinate system is therefore

$$dv = \frac{2^{r_1-r_2-1+\delta_{r_1^0}} R}{Y_0^2} \bigwedge_{i=0}^{r-1} dY_i \wedge dx.$$

We now verify that the main term appearing in $I_\phi(T)$ during the contour shift in lemma 12 and equation (47) is in fact $\langle E(z|g), 1 \rangle / \text{Vol}(Y) \langle \phi F_k, F_k \rangle$. The residue is equal to

$$V_c^{-1} G(-1, 0) \text{Res}_{s=1} \phi(s) \langle \phi F_k, F_k \rangle,$$

and it may easily be seen that $\langle E(z|g), 1 \rangle = 2^{-r_2} \omega_+^{-1} \sqrt{|D|} G(-1, 0)$. Therefore to show that the two expressions are equal we only need to show that $\text{Vol}(Y) = V_c 2^{-r_2} \omega_+^{-1} \sqrt{|D|} (\text{Res}_{s=1} \phi(s))^{-1}$. This follows easily from the standard method of computing the volume of fundamental domains using Eisenstein series, and substituting the value of $\text{Res}_{s=1} \phi(s)$ gives

$$\text{Vol}(Y) = \frac{2^{-4r_2+1} |D|^{3/2} \zeta_F(2)}{\pi^n}.$$

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